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BOUNDARY PROBLEMS IN NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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PREFACE

Last year I gave a series of Extension Lectures on non-linear partial differential equations at the Lucknow University. These have now been written down in the form of a monograph to be published by the University. This small volume aims at giving a preliminary account of the methods employed and the results achieved for non-linear equations of the standard types. The first two chapters contain generalisations of my previous results, and are being published now for the first time. These lectures were originally advertised under the heading "The Theory of Non-Linear partial Differential Equations" but the present title has been considered more appropriate for the monograph.

I take this opportunity of offering my grateful thanks to Professor Birbal Sahni, F.R.S., for kindly arranging for the lectures in Lucknow, and to Dr. A. N. Singh, D.Sc., for his kind help during the printing of the book.

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INTRODUCTION

The theory of the boundary value problems in linear partial differential equations has been extensively studied, and the results incorporated in a large number of memoirs and treatises which have been reported in the Enzyklópädie articles* by Sommerfeld and Lichtenstein.

The non-linear partial differential equations were, however, neglected for a long time, owing probably to the fact that not many physical phenomena were known which gave rise to boundary problems in such equations. But the situation is changing rapidly, and modern explanations of various physical phenomena are yielding more and more non-linear equations. This fact has induced several investigators to take up the subject

* A. Sommerfeld: "Randwertaufgaben in der Theorie der partiellen Differentialgleichungem": Enzykl. d. Math. Wiss. II A. 7c, pp. 504-570.

L. Lichtenstein: "Neuere Entwicklung der Theorie partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus": Enzykl. d. Math. Wiss. II c. 12 pp. 1277-1334.

systematically, and thus two different methods of attack have been evolved.

The first method is that of Green's function well known in the potential theory. This method was developed by M. Émile Picard in 1890, and applied by him to linear as well as non-linear elliptic and hyperbolic equations.

In 1907 Holmgren and Levi simultaneously developed a similar method for the linear parabolic equations, employing, of course, the newly discovered theory of integral equations. Inspired by the works of Picard and Levi, M. Maurice Gevrey made a fairly complete investigation of the parabolic equations in 1913-1914.

The method of Green's function has been well developed and applied to non-linear equations in a large number of memoirs and treatises, a selection of which is given in the bibliography.

Now in physical theories as well as in mathematical analysis, the solution of a boundary value problem in the form of a Fourier series or of a series of eigenfunctions is of a special value. In fact, since the classical work of Fourier himself the linear problems have been often solved in this way. Lichtenstein was probably the first

writer to give the solution of the non-linear hyperbolic equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = u^2$$

in the form of a fourier series*

$$u(x, t) = \sum_n v_n(t) \sin nx.$$

At his suggestion the present writer undertook a detailed study of the non-linear parabolic and hyperbolic equations from this point of view. A few results have been published in papers mentioned in the Bibliography.

A fairly generalised theory is being presented in these lectures. The first chapter gives the so called "Fourier Method" for the non-linear parabolic equation :

$$\frac{\partial}{\partial x}\left\{p(x)\frac{\partial u}{\partial x}\right\} = P\left(x, t, u, \frac{\partial u}{\partial x}\right).$$

The solution of the first boundary problem is given in the form of a series of Sturm-Liouville eigenfunctions, whose asymptotic expansion plays a fundamental part in the treatment. The Fourier coefficients are determined with the help of an

* Unless otherwise stated, the summation is always to be taken from 1 to ∞ throughout this book.

infinite system of non-linear integral equations which is solved by the method of successive approximations.

The method is then shown to hold for equations of higher order and for elliptic-parabolic equations. It is further shown that the problems for non-rectangular domains and for mixed boundary values can be solved by reducing them to the above problem with the help of suitable transformations of variables. Systems of differential equations can also be treated in this way. [Cf. Siddiqi, (7)].

The second chapter gives the Fourier method for non-linear hyperbolic equations. The method is similar to that for the parabolic equations, but there is a fundamental difference. The parabolic equations have a unique solution in Fourier series for any domain arbitrarily large, whereas the hyperbolic equations have a solution for a sufficiently small domain. This is compensated by the fact that the boundary values for parabolic equations are more restricted than those for the hyperbolic equations. The generalisation for non-vanishing boundary values is given in the last section, though other generalisations for non-rectangular domains, for mixed boundary problems, for equations of

higher order etc., have been left out, their treatment being completely analogous to that for the parabolic equations.

Many problems have still to be considered both for the parabolic and hyperbolic equations.* For instance, if the right hand member of the given differential equation contains the derivative $\frac{\partial u}{\partial t}$, then the system of integral equations which determines the Fourier coefficients is replaced by a system of integro-differential equations, for which a method of solution has yet to be developed. Similarly, the second boundary problem in which the derivative $\frac{\partial u}{\partial x}$ vanishes both at $x = 0$ and at $x = \pi$ still remains to be considered.

The account for non-linear elliptic equations in Chapter III is taken from the classical memoir of M. Picard published in the Journal de Mathematique in 1890. Similarly the account for non-linear parabolic equations in Chapter IV is taken from M. Gevrey's paper published in the same

*During the time the book was in the press, these problems have been dealt with by S. M. Sundaram, [1], [2], [3], working at Madras and Hyderabad. [See Chap. V].

Journal in 1913. It will be seen from these Chapters that the method of Green's function establishes the existence of the solution for sufficiently small domains.

A resumé of the results for linear elliptic and parabolic equations is given in the opening sections of the respective chapters. Goursat's "Cours d' Anlyse Mathématique" vol. 3, or Frank-von Mises' "Differential-und-Integral gleichungen der Mathematischen Physik" vol. 1 can be consulted for proofs and other details of these results.

No account is given of the method of Green's function for the hyperbolic equations because it is almost similar to the method for elliptic and parabolic equations. Besides, an almost exhaustive account of hyperbolic equations is given in Courant and Hilhert's admirable book "Methoden der mathematischen Physik" vol. 2, recently published.

CHAPTER I

NON-LINEAR PARABOLIC EQUATIONS

1·1. Convergence of an Infinite Series

Let $p(x)$ be a function > 1 in the interval $0 \leqslant x \leqslant \pi$, and let $p(x)$ and $\frac{dp}{dx}$ be continuous and uniformly bounded in this interval:

$$\left|p(x)\right| \leqslant M, \quad \left|\frac{dp}{dx}\right| \leqslant M,$$

M being an absolute Constant.

Let the Sturm—Liouville equation

$$\frac{d}{dx}\left\{p(x)\frac{d\phi}{dx}\right\} + \lambda\phi = 0, \quad \ldots\ldots\ldots\ldots \quad (2)$$

for the boundary conditions

$$\phi(0) = 0, \ \phi(\pi) = 0 \quad \ldots\ldots\ldots\ldots \quad (3)$$

have the eigenvalues λ_n and the eigenfunctions $\phi_n(x)$, where $n = 1, 2, \ldots \infty$. We assume that the set of functions $\phi_n(x)$ is orthonormal. The asymptotic expansions of λ_n and $\phi_n(x)$ are known* to be

*Curant—Helbert: "Methoden der Mathematischen Physik", Vol. I, Chap. 5, § 11, p. 291.

$$\left.\begin{aligned}
&\lambda_n = n^2\frac{\pi^2}{a^2} + O(1),\ a = \int_0^\pi \frac{1}{\sqrt{p(x)}}\,dx,\\
&\phi_n(x) = c_n\frac{\sin nq}{p^{1/4}} + O\left(\frac{1}{n}\right),\ q(x) = \frac{\pi}{a}\int_0^x \frac{dx}{\sqrt{p(x)}},\\
&\frac{d\phi_n}{dx} = c_n\frac{n\pi}{a}\frac{\cos nq}{p^{1/5}} + O(1),\ \frac{1}{c_n^2} = \int_0^\pi \frac{\sin^2 nq}{\sqrt{p(x)}}\,dx.
\end{aligned}\right\} \quad (4)$$

Evidently $\dfrac{1}{c_n^2}$ does not approach zero whatever n may *be*, so that $\dfrac{1}{c_n}$ is different from zero, and therefore c_n bounded for all n. Consequently, $\phi_n(x)$ and $\dfrac{1}{n}\dfrac{d\phi_n}{dx}$ are also uniformly bounded.

Hence we can write for all x in $0 \leqslant x \leqslant \pi$

$$\left|\phi_n(x)\right| \leqslant b,\ \left|\frac{1}{n}\frac{d\phi_n}{dx}\right| \leqslant b, \qquad \ldots\ldots\ldots \quad (5)$$

$$(n = 1, 2, \ldots)$$

where b is an absolute constant.

Finally, let the function $p_{r,s}(x, t)$ be defined in the domain $0 \leqslant x \leqslant \pi$, $0 \leqslant t < \infty$ for any $r,s \geqslant 1$, and suppose that $p_{rs}(x, t)$, $\dfrac{\partial p_{r,s}}{\partial x}$, $\dfrac{\partial^2 p_{r,s}}{\partial x^2}$ are uniformly bounded in the whole domain for all $r, s \geqslant 1$:

$$\left| p_{r,s}(x, t) \right| \leqslant N, \left| \frac{\partial p_{r,s}}{\partial x} \right| \leqslant N, \left| \frac{\partial^2 p_{r,s}}{\partial x^2} \right| \leqslant N, \quad (6)$$

N being also an absolute constant.

Consider the function

$$a_n = \int_0^\pi p_{r,s}(x,t)\phi_{k_1}(x)\ldots\phi_{k_r}(x)\phi'_{l_1}(x)\ldots\phi'_{l_s}(x)\phi_n(x)dx, \quad (7)$$

where n, k , ..., k_r, l_1, ..., l_s, r, s can take all positive integral values from 1 to ∞, and where dashes are used to denote differentiation *w. r.* to x.

We shall establish the following result:

For all $k_1, \ldots, k_r, l_1, \ldots, l_s \geqslant 1$ we have

$$\sum_n \frac{\sqrt{\lambda_n}\, |a_n|}{(\lambda_{k_1} \ldots \lambda_{k_r})(\lambda_{l_1} \ldots \lambda_{l_s})^{3/2}} < c\, b^{r+s}\, q_1(r)\, q_2(s), \quad (8)$$

where c is an absolute constant independent of r, s, and $q_1(r)$, $q_2(s)$ are polynomials in r and s respectively.

To prove this, we integrate (7) by parts, and get

$$a_n = -\frac{1}{\lambda_n}\int_0^\pi p_{r,s}(x,t)\,\phi_{k_1}\cdots\phi_{k_r}\,\phi'_{l_1}\cdots\phi'_{l_s}\frac{d}{dx}(p\phi'_n)dx,$$

$$= -\frac{1}{\lambda_n}\int_0^\pi \phi_n\frac{d}{dx}\left\{p\,\frac{d}{dx}\left(p_{r,s}\,\phi_{k_1}\cdots\phi_{k_r}\,\phi_{l_1}\cdots\phi'_{l_s}\right)\right\}dx. \quad \text{.........} \quad (9)$$

But we have

$$p\frac{d}{dx}\left(p_{r,s}\,\phi_{k_1}\cdots\phi_{k_r}\,\phi'_{l_1}\cdots\phi'_{l_s}\right)$$

$$=pp_{r,s}\frac{d}{dx}\left(\phi_{k_1}\cdots\phi_{k_r}\right)\phi'_{l_1}\cdots\phi'_{l_s}$$

$$+\,pp_{r,s}\,\phi_{k_1}\cdots\phi_{k_r}\frac{d}{dx}\left(\phi'_{l_1}\cdots\phi'_{l_s}\right)$$

$$+\,pp'_{r,s}\,\phi_{k_1}\cdots\phi_{k_r}\,\phi'_{l_1}\cdots\phi'_{l_s}$$

$$=p_{r,s}\,\phi'_{l_1}\cdots\phi'_{l_s}\sum_{j=1}^{r}\left\{\phi_{k_1}\cdots\phi_{k_{j-1}}\left(p\phi'_{k_j}\right)\phi_{k_{j+1}}\cdots\phi_{k_r}\right\}$$

$$+\,p_{r,s}\,\phi_{k_1}\cdots\phi_{k_r}\sum_{j=1}^{s}\left\{\phi'_{l_1}\cdots\phi'_{l_{j-1}}\left(p\phi''_{l_j}\right)\phi'_{l_{j+1}}\cdots\phi'_{l_s}\right\}$$

$$+\,pp'_{r,s}\,\phi_{k_1}\cdots\phi_{k_r}\,\phi'_{l_1}\cdots\phi'_{l_s}. \qquad \text{.........} \quad (10)$$

We must now differentiate these expressions. Thus

$$\frac{d}{dx}\left[p_{r,s}\,\phi'_{l_1}\cdots\phi'_{l_s}\sum_{j=1}^{r}\left\{\phi_{k_1}\cdots\phi_{k_{j-1}}\left(p\phi'_{k_j}\right)\phi_{k_{j+1}}\cdots\phi_{k_r}\right\}\right]$$

$$=p'_{r,s}\,\phi'_{l_1}\cdots\phi'_{l_s}\sum_{j=1}^{r}\left\{\phi_{k_1}\cdots\phi_{k_{j-1}}\left(p\phi'_{k_j}\right)\phi_{k_{j+1}}\cdots\phi_{k_r}\right\}$$

$$+\,p_{r,s}\sum_{j=1}^{r}\left\{\phi_{k_1}\cdots\phi_{k_{j-1}}\left(p\phi'_{k_j}\right)\phi_{k_{j+1}}\cdots\phi_{k_r}\sum_{i=1}^{s}\phi'_{l_1}\cdots\phi'_{l_{i-1}}\right.$$

$$\left.\phi''_{l_i}\,\phi'_{l_{i+1}}\cdots\phi'_{l_s}\right\}$$

$$+ p_{r,s}\,\phi'_{l_1}\cdots\phi'_{l_s}\sum_{j=1}^{r}\left\{\phi_{k_1}\cdots\phi_{k_{j-1}}\frac{d}{dx}\left(p\phi'_{k_j}\right)\phi_{k_{j+1}}\cdots\phi_{k_r}\right\}$$

$$+ p_{r,s}\,\phi'_{l_1}\cdots\phi'_{l_s}\sum_{j=1}^{r}\left\{p\phi'_{k_j}\sum_{i=1}^{r}{}'\left(\phi_{k_1}\cdots\phi_{k_{i-1}}\phi'_{k_i}\phi_{k_{i+1}}\cdots\phi_{k_r}\right)\right\}$$

$$= p'_{r,s}\,\phi'_{l_1}\cdots\phi'_{l_s}\sum_{j=1}^{r}\left\{\phi_{k_1}\cdots\phi_{k_{j-1}}\left(p\phi'_{k_j}\right)\phi_{k_{j+1}}\cdots\phi_{k_r}\right\}$$

$$- p_{r,s}\sum_{j=1}^{r}\left\{\phi_{k_1}\cdots\phi_{k_{j-1}}\phi'_{k_j}\phi_{k_{j+1}}\cdots\phi_{k_r}\sum_{i=1}^{s}\left[\phi'_{l_1}\cdots\phi'_{l_{i-1}}\right.\right.$$

$$\left.\left.\left(\lambda_{l_i}\phi_{l_i}+p'\,\phi'_{l_i}\right)\times\,\phi'_{l_{i+1}}\cdots\phi'_{l_s}\right]\right\}$$

$$- p_{r,s}\,\phi_{k_1}\cdots\phi_{r}\,\phi'_{l_1}\cdots\phi'_{l_s}\left(\lambda_{k_1}+\lambda_{k_2}+\ldots+\lambda'_{kr}\right)$$

$$+ pp_{r,s}\,\phi'_{l_1}\cdots\phi'_{l_s}\sum_{i,\,j=1}^{r}{}'\left(\phi_{k_1}\cdots\phi_{k_{i-1}}\phi'_{k_i}\phi_{k\,i+1}\cdots\right.$$

$$\left.\phi_{k_{j-1}}\phi'_{k_j}\phi_{kj+1}\cdots\phi_{k_r}\right), \qquad \ldots\ldots\ldots \quad (11)$$

where Σ' means that the term $i = j$ is omitted.

We see that the number of terms on the right of (11) is

$$= r + 2rs + r + r\,(r - 1) = r(r + 2s + 1). \quad (12)$$

We differentiate now the second line on the right of (10). On account of the relation $p\phi''_{l_j} = -\,(\lambda_{l_j} + p'\,\phi'_{l_j})$, this term is

$$= - p_{r,s}\, \phi_{k_1} \cdots \phi_{k_r} \sum_{j=1}^{s} \left(\lambda_{l_j}\, \phi'_{l_1} \cdots \phi'_{l_{j-1}}\, \phi_{l_j} \phi'_{l_{j+1}} \cdots \phi'_{l_s} \right)$$

$$- s p' p_{r,s}\, \phi_{k_1} \cdots \phi_{k_r} \phi'_{l_1} \cdots \phi'_{l_s}. \qquad \ldots\ldots \quad (13)$$

On differentiating the first line in (13) we get

$$\frac{d}{dx}\left\{ - p_{r,s}\, \phi_{k_1} \cdots \phi_{k_r} \sum_{j=1}^{s} \left(\lambda_{l_j}\, \phi'_{l_1} \cdots \phi'_{l_{j-1}}\, \phi_{l_j}\, \phi'_{l_{j+1}} \cdots \phi'_{l_s} \right) \right\}$$

$$= - p'_{r,s}\, \phi_{k_1} \cdots \phi_{k_r} \sum_{j=1}^{s} \left(\lambda_{l_j}\, \phi'_{l_1} \cdots \phi'_{l_{j-1}}\, \phi_{l_j}\, \phi'_{l_{j+1}} \cdots \phi'_{l_s} \right)$$

$$- p_{r,s} \sum_{j=1}^{s} \left\{ \lambda_{l_j}\, \phi'_{l_1} \cdots \phi'_{l_{j-1}}\, \phi_{l_j}\, \phi'_{l_{j+1}} \cdots \phi'_{l_s} \sum_{i=1}^{r} \left(\phi_{k_1} \cdots \right.\right.$$

$$\left.\left. \phi_{k_{i-1}}\, \phi'_{k_i}\, \phi_{k_{i+1}} \cdots \phi_{k_r} \right) \right\}$$

$$- p_{r,s}\, \phi_{k_1} \cdots \phi_{k_r}\, \phi'_{l_1} \cdots \phi'_{l_s} \left(\lambda_{l_1} + \cdots + \lambda_{l_s} \right)$$

$$- p_{r,s}\, \phi_{k_1} \cdots \phi_{k_r} \sum_{j=1}^{s} \left\{ \lambda_{l_j}\, \phi_{l_j} \sum_{i=1}^{s}{}' \left(\phi'_{l_1} \cdots \phi'_{l_{i-1}}\, \phi''_{l_i} \right.\right.$$

$$\left.\left. \phi'_{l_{i+1}} \cdots \phi'_{l_s} \right) \right\} \qquad \ldots\ldots \quad (14)$$

The last term on the right of (14) is equal to

$$\frac{p_{r,s}}{p}\, \phi_{k_1} \cdots \phi_{k_r} \sum_{j=1}^{s} \left\{ \lambda_{l_j}\, \phi_{l_j} \sum_{i=1}^{s}{}' \left[\phi'_{l_1} \cdots \phi'_{l_{i-1}} \left(\lambda_{l_i} \phi_{l_i} + p' \phi'_{l_i} \right) \right.\right.$$

$$\left.\left. \phi'_{l_{i+1}} \cdots \phi'_{l_s} \right] \right\} \ldots\ldots \quad (15)$$

The number of terms on the right of (14) is therefore

$$= s + sr + 2s(s-1) + s = s\,(r + 2s). \qquad (16)$$

We differentiate now the second line in (13), and get

$$\frac{d}{dx}\left(-\,sp'\,p_{r,s}\,\phi_{k_1}\cdots\phi_{k_r}\,\phi'_{l_1}\cdots\phi'_{l_s}\right)$$

$$= -\,sp''\,p_{r,s}\,\phi_{k_1}\cdots\phi_{k_r}\,\phi'_{l_1}\cdots\phi'_{l_s} - sp'\,p'_{r,s}\,\phi_{k_1}\cdots$$
$$\phi_{k_r}\,\phi'_{l_1}\cdots\phi'_{l_s}$$

$$-\,sp'\,p_{r,s}\,\phi'_{l_1}\cdots\phi'_{l_s}\sum_{j=1}^{r}\left(\phi_{k_1}\cdots\phi_{k_{j-1}}\,\phi'_{k_j}\,\phi_{k_{j+1}}\cdots\phi_{k_r}\right)$$

$$-\,sp'\,p_{r,s}\,\phi_{k_1}\cdots\phi_{k_r}\sum_{j=1}^{s}\left(\phi'_{l_1}\cdots\phi'_{l_{j-1}}\,\phi''_{l_j}\right.$$
$$\left.\phi'_{l_{j+1}}\cdots\phi'_{l_s}\right) \quad \ldots\ldots \quad (17)$$

The last term on the right of (17) is equal to

$$s\,\frac{p'p_{r,s}}{p}\,\phi_{k_1}\cdots\phi_{k_r}\sum_{j=1}^{s}\left\{\phi'_{l_1}\cdots\phi'_{l_{j-1}}\left(\lambda_{l_j}\,\phi_{l_j} + p'\,\phi'_{l_j}\right)\right.$$
$$\left.\phi'_{l_{j+1}}\cdots\phi'_{l_s}\right\} \quad \ldots\ldots \quad (18)$$

Thus the number of terms on the right of (17) is

$$= s(2 + r + 2s). \qquad \ldots\ldots\ldots\ldots \quad (19)$$

Finally we differentiate the last line on the right of (10), and get

$$\frac{d}{dx}\left(pp'_{r,s}\phi_{k_1}\cdots\phi_{k_r}\phi'_{l_1}\cdots\phi'_{l_s}\right)$$

$$=p'p'_{r,s}\phi_{k_1}\cdots\phi_{k_r}\phi'_{l_1}\cdots\phi'_{l_s}+pp''_{r,s}\phi_{k_1}\cdots\phi_{k_r}\phi'_{l_1}\cdots\phi'_{l_s}$$

$$+\,pp'_{r,s}\phi'_{l_1}\cdots\phi'_{l_s}\sum_{j=1}^{r}\left(\phi_{k_1}\cdots\phi_{k_{j-1}}\phi'_{k_j}\phi_{k_{j+1}}\cdots\phi_{k_r}\right)$$

$$+\,pp'_{r,s}\phi_{k_1}\cdots\phi_{k_r}\sum_{j=1}^{s}\left(\phi'_{l_1}\cdots\phi'_{l_{j-1}}\phi''_{l_j}\phi'_{l_{j+1}}\cdots\phi'_{l_s}\right).\quad(20)$$

The last term on the right of (20) is equal to

$$-\,p_{r,s}\phi_{k_1}\cdots\phi_{k_r}\sum_{j=1}^{s}\Big\{\phi'_{l_1}\cdots\phi'_{l_{j-1}}\left(\lambda_{l_j}\phi_{l_j}+p'\phi'_{l_j}\right)$$

$$\phi'_{l_{j+1}}\cdots\phi'_{l_s}\Big\}.\qquad\ldots\ldots\ldots\quad(21)$$

Thus the number of terms on the right of (20) is

$$=2+r+2s.\qquad\ldots\ldots\ldots\quad(22)$$

Therefore the integrand in (9) has altogether m terms where on account of (12), (16), (19) and (22) we find

$$m=r(r+2s+1)+s(r+2s)+s(2+r+2s)$$
$$+\,(2+r+2s)=(r+2s+1)^2+1\qquad\ldots\ldots\quad(23)$$

Moreover, each term in the integrand contains

(*a*) either $p_{r,s}(x,t)$ or $p'_{r,s}(x,t)$ or $p''_{r,s}(x,t)$;

(*b*) either $p(x)$ or $p'(x)$ or $\frac{p'}{p}$ or $\frac{p'^2}{p}$ or 1;

(*c*) either λ_{k_j} or 1 $(j = 1, 2, \ldots r)$;

(*d*) either λ_{l_j} or 1 $(j = 1, 2, \ldots s)$;

(*e*) either $\phi_{k_j}(x)$ or $\phi'_{k_j}(x)$ $(j = 1, 2, \ldots r)$;

(*f*) either $\phi_{l_j}(x)$ or $\phi'_{l_j}(x)$ $(j = 1, 2, \ldots s)$;

(*g*) $\phi_n(x)$.

To make the result independent of $k_1, \ldots, k_r$ we must divide the expression by $\lambda_{k_1} \ldots \lambda_{k_r}$. But one and the same term may contain λ_{l_j} and ϕ'_{l_j} together, therefore it is not sufficient to divide by λ_{l_j} only, but by $\lambda_{l_j}^{3/2}$. Thus the whole expression must be divided by $(\lambda_{l_1} \ldots \lambda_{l_s})^{3/2}$. Then each term in the integrand will be uniformly bounded for all $n, k_1, \ldots, k_r, l_1, \ldots, l_s \geq 1$.

We shall denote the m integrals thus arising by

$$a_n^{(1)},\ a_n^{(2)},\ \ldots,\ a_n^{(m)}. \qquad \ldots\ldots \quad (24)$$

From (9) we obtain then

$$\frac{\sqrt{\lambda_n}\,|a_n|}{(\lambda_{k_1} \ldots \lambda_{k_r})(\lambda_{l_1} \ldots \lambda_{l_s})^{3/2}} < \frac{1}{\sqrt{\lambda_n}} \sum_{i=1}^{m} |a_n^{(i)}|. \qquad \ldots\ldots \quad (25)$$

From Parseval's theorem for the Sturm-Liouville eigenfunctions we know that the series

$\sum\limits_n (a_n^{(i)})^2$ is convergent for all $i = 1, 2, \ldots, m$, so that

$$\sum_n (a_n^{(i)})^2 < \pi N^2 M^4 b^{2(r+s)}. \qquad \ldots\ldots \quad (26)$$

From (25) we obtain on summing over n and applying Schwarz's inequality as well as the inequality $2ab \leqslant a^2 + b^2$:

$$\left\{\sum_n \frac{\sqrt{\lambda_n}|a_n|}{(\lambda_{k_1} \ldots \lambda_{k_r})(\lambda_{l_1} \ldots \lambda_{l_s})^{3/2}}\right\}^2 < \sum_n \frac{1}{\lambda_n} \cdot \sum_n \left\{\sum_{i=1}^m |a_n^{(i)}|\right\}^2.$$

$$< 2 \sum_n \frac{1}{\lambda_n} \cdot \sum_{i=1}^m \left\{\sum_n \left(a_n^{(i)}\right)^2\right\}$$

$$< 2 \cdot \frac{\pi^2}{6} \cdot \pi N^2 M^4 b^{2(r+s)} m, \qquad \ldots\ldots \quad (27)$$

where we have used the result that

$$\sum_n \frac{1}{\lambda_n} < \sum_n \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Writing

$$c^2 = \frac{\pi^3}{3} N^2 M^4, \qquad \ldots\ldots \quad (28)$$

and

$$\left.\begin{aligned} q_1(r) &= r + 1, \\ q_2(s) &= 2s + 1, \end{aligned}\right\}, \qquad \ldots\ldots \quad (29)$$

the truth of (8) becomes obvious.

1·2. An Infinite System of Non-linear Integral Equations.

Let the numbers n, $k_1, \ldots, k_r$, $l_1, \ldots, l_s$, r, s

take as before all positive integral values from 1 to ∞, and let $f_n(x)$, $g_n(x, y)$, $h_n(y, k_1, \ldots, k_r, l_1, \ldots, l_s\, r, s)$ be sequences of functions, denoted simply by f_n, g_n, h_n, defined for all values of x, y:

$$0 \leqslant x < \infty \,;\; 0 \leqslant y \leqslant x < \infty, \quad \ldots \quad (1)$$

and satisfying the relations

$$\Sigma \,|\, f_n(x) \,| \leqslant c \text{ for all } x \text{ in (1)} \,; \quad \ldots\ldots \quad (2)$$

$$\left| \int_0^x g_n(x, y)\, dy \right| \leqslant B \text{ for all } x \text{ in (1) and all } n \geqslant 1; \quad (3)$$

$\sum_n |\, h_n \,| < A\, b^{r+s}\, q_1(r)\, q_2(s)$ for all y in (1) and all k's and l's $\geqslant 1$. (4)

A, B, b, c are absolute constants, and $q_1(r)$, $q_2(s)$ denote polynomials in r and s respectively.

We consider the infinite system of non-linear integral equations:

$$u_n(x) = f_n(x) + \int_0^x g_n(x, y) \sum_{r, s} \sum_{\substack{k_1, \ldots, k_r \\ l_1, \ldots\ldots, l_s}} h_n u_{k_1}(y) \ldots$$

$$u_{k_r}(y) \,.\, u_{l_1}(y) \ldots u_{l_s}(y)\, dy, \quad \ldots\ldots \quad (5)$$

$$(n = 1, 2, \ldots \infty)$$

We shall prove that one and only solution of the system (5) exists for all values of x, provided only that c is less than a certain constant.

We shall solve the system (5) by the method of successive approximations, and write for this purpose

$$u_n^{(0)}(x) = f_n(x), \qquad \ldots\ldots \quad (6)$$

and for all $m \geqslant 1$

$$u_n^{(m)}(x) = f(x) + \int_0^x g_n(x, y) \sum_{r, s} \sum_{\substack{k_1, \ldots, k_r \\ l_1, \ldots, l_s}} h_n u_{k_1}^{(m-1)}(y) \ldots$$

$$u_{k_r}^{(m-1)}(y) u_{l_1}^{(m-1)}(y) \ldots u_{l_s}^{(m-1)}(y)\, dy. \qquad \ldots\ldots \quad (7)$$

We must prove that these approximations are convergent, i.e., that $\lim_{m \to \infty} u_n^{(m)}(x)$ exists for all $n \geqslant 1$. This will be the case if the doubly infinite series $\sum_{m=0}^{\infty} \sum_n |u_n^{(m+1)}(x) - u_n^{(m)}(x)|$ is uniformly convergent for all x.

We begin by proving that the series

$$\sum_n |u_n^{(m)}(x)| \qquad \ldots\ldots \quad (8)$$

is uniformly convergent for all x and all m.

Summing (7) formally over n, and taking account of (2), (3) and (4), we obtain for all x

$$\sum_n |u_n^{(m)}(x)| < c + a \sum_r q_1(r) \{b \operatorname{Max} \sum_n |u_n^{(m-1)}(x)|\}^r \times$$

$$\sum_s q_2(s) \{b \operatorname{Max} \sum_n |u_n^{(m-1)}(x)|\}^s, \qquad \ldots\ldots \quad (9)$$

where we have written $a = AB$.

Since $q_1(r)$ and $q_2(s)$ are polynomials, two constants d and e* can be found such that

$$q_1(r) \leqslant d^r \text{ for all } r \geqslant 1, \; q_2(s) \leqslant e^s \text{ for all } s \geqslant 1 \qquad (10)$$

From (6) we get for all $x \geqslant 0$

$$\sum_n | u_n^{(0)}(x) | = \sum_n | f_n(x) | \leqslant c. \quad \ldots \quad (11)$$

Substituting (10) and (11) in (9) for $m = 1$, we get

$$\sum_n |u_n^{(1)}(x)| < c + a \left\{ \sum_r (bcd)^r . \sum_s (bce)^s \right\}.$$

Let $\qquad c < \text{Min}\left(\frac{1}{bd}, \frac{1}{be}\right)$, then

$$\sum_n |u_n^{(1)}(x)| < c + a \frac{bcd}{1 - bcd} . \frac{bce}{1 - bce} .$$

If c is taken to be less than

$$\frac{2}{ab^2de + bd + be}, \text{ then } a \frac{bcd}{1 - bcd} \frac{bce}{1 - bce} < c,$$

and we obtain

$$\sum_n | u_n^{(1)}(x) | < c + c = 2c. \quad \ldots\ldots \quad (12)$$

Writing $m = 2$ in (9), and substituting (12), we get

$$\sum_n u_n^{(1)}(x) < c + a \left\{ \sum_r (2bcd)^r \sum_s (2bce)^s \right\}.$$

*e must not be confused with the base of the exponential functions.

Assuming that $c < \operatorname{Min}\left(\frac{1}{2bd}, \frac{1}{2be}\right)$, we find

$$\sum_n u_n^{(2)}(x) < c + a\,\frac{2bcd}{1 - 2bcd}\,\frac{2bce}{1 - 2bce}.$$

Supposing that

$$c < \frac{1}{2b^2de + bd + be}, \qquad \ldots\ldots \quad (13)$$

which makes $a\,\frac{2bcd}{1 - 2bcd}\,\frac{2bce}{1 - 2bce} < c$, we obtain

$$\sum_n |u_n^{(2)}(x)| < c + c = 2c.$$

Thus we find generally for all $m \geqslant 1$ and all $x \geqslant 0$

$$\sum_n |u_n^{(m)}(x)| < 2c, \qquad \ldots\ldots \quad (14)$$

establishing the uniform convergence of the series (8).

We shall prove now that the double series

$$\sum_{m=0}^{\infty} \sum_n |u_n^{(m+1)}(x) - u_n^{(m)}(x)| \qquad \ldots\ldots \quad (15)$$

is uniformly convergent for all $x \geqslant 0$.

In fact we have from (7)

$$u_n^{(m+1)}(x) - u_n^{(m)}(x) = \int_0^x g_n(x, y) \sum_{rs} \sum_{\substack{k_1, \ldots, k_r, \\ l_1, \ldots, l_s}} b_n$$

$$\times \left\{ u_{k_1}^{(m)}(y) \ldots u_{l_s}^{(m)}(y) - u_{k_1}^{(m-1)}(y) u_{k_2}^{(m-1)}(y) \ldots u_{l_s}^{(m-1)}(y) \right\} dy$$

$$
\begin{aligned}
= \int_0^x g_n(x, y) \sum_{rs} \sum_{\substack{k_1, \ldots, k_r \\ l_1, \ldots, l_s}} & h_n \Big\{ u_{k_1}^{(m)}(y) \ldots u_{l_{s-1}}^{(m)}(y) \\
& \qquad \left(u_{l_s}^{(m)}(y) - u_{l_s}^{(m-1)}(y) \right) \\
& + u_{l_s}^{(m-1)}(y)\, u_{k_1}^{(m)}(y) \ldots u_{l_{s-2}}^{(m)}(y) \left(u_{l_{s-1}}^{(m)}(y) - u_{l_{s-1}}^{(m-1)}(y) \right) \\
& + u_{l_{s-1}}^{(m-1)}(y)\, u_{l_s}^{(m-1)}(y)\, u_{k_1}^{(m)}(y) \ldots u_{l_{s-3}}^{(m)}(y) \left(u_{l_{s-2}}^{(m)}(y) \right. \\
& \qquad \left. - u_{l_{s-2}}^{(m-1)}(y) \right) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + u_{k_2}^{(m-1)}(y)\, u_{k_3}^{(m-1)}(y) \ldots u_{l_s}^{(m-1)}(y) \left(u_{k_1}^{(m)}(y) \right. \\
& \qquad \left. - u_{k_1}^{(m-1)}(y) \right) \Big\} dy.
\end{aligned}
$$

The uniform convergence of $\sum\limits_n \left| u_n^{(m+1)}(x) - u_n^{(m)}(x) \right|$ is seen therefore to depend on that of $\sum\limits_n \left| u_n^{(m)}(x) - u_n^{(m-1)}(x) \right|$, so that on account of (14) we obtain for all x

$$
\begin{aligned}
\sum_n \left| u_n^{(m+1)}(x) - u_n^{(m)}(x) \right| < & a \sum_{r,s} (r+s)\, b^{r+s} (2c)^{r+s-1} \\
& q_1(r)\, q_2(s) \times \operatorname{Max} \sum_n \left| u_n^{(m)}(x) - u_n^{(m-1)}(x) \right| \\
& < a \sum_r (r+1)\, b^r (2c)^{r-1} q_1(r) \times \sum_s (s+1)\, b^s \\
& (2c)^s q_2(s) \times \operatorname{Max} \sum_n \left| u_n^{(m)}(x) - u_n^{(m-1)}(x) \right| \\
& \ldots\ldots\ldots\ldots\ldots \quad (16)
\end{aligned}
$$

But we can find two numbers d_1 and e_1 such that

$$\left.\begin{aligned}(r+1)\,q_1(r) &\leqslant d_1^r \text{ for all } r \geqslant 0\\ (s+1)\,q_2(s) &\leqslant e_1^s \text{ for all } s \geqslant 0\end{aligned}\right\} \quad \ldots\ldots \quad (17)$$

The relation (16) then becomes

$$\sum_n \left| u_n^{(m+1)}(x) - u_n^{(m)}(x) \right| < ab \sum_{r=0}^{\infty} (2bcd_1)^r . \sum_s (2bce_1)^s$$
$$\times \operatorname{Max} \sum_n \left| u_n^{(m)}(x) - u_n^{(m-1)}(x) \right| . \quad \ldots\ldots \quad (18)$$

If we assume that $c < \operatorname{Min}\left(\frac{1}{2bd_1}, \frac{1}{2be_1}\right)$, we get

$$\sum_n \left| u_n^{(m+1)}(x) - u_n^{(m)}(x) \right| < ab \frac{1}{(1-2bcd_1)} \frac{2bce_1}{(1-2bce_1)}$$
$$\times \operatorname{Max} \sum_n \left| u_n^{(m)}(x) - u_n^{(m-1)}(x) \right| . \quad \ldots\ldots \quad (19)$$

Finally we assume that

$$c < \frac{1}{b(abe_1 + d_1 + e_1)}, \quad \ldots\ldots \quad (20)$$

which gives

$$\gamma = \frac{2ab^2 ce_1}{(1-2bcd_1)(1-2bce_1)} < 1 \quad \ldots\ldots \quad (21)$$

Repeating the reduction formula (19) m times, we obtain

$$\sum_n \left| u_n^{(m+1)}(x) - u_n^{(m)}(x) \right|$$
$$< \gamma^m \operatorname{Max} \sum_n \left| u_n^{(1)}(x) - u_n^{(0)}(x) \right| . \quad \ldots\ldots \quad (22)$$

Summing over m from o to ∞, we get for all $x \geqslant 0$

$$\sum_{m=0}^{\infty} \sum_{n} \left| u_n^{(m+1)}(x) - u_n^{(m)}(x) \right| \leqslant \sum_{m} \gamma^m \operatorname{Max} \sum_{n} \left| u_n^{(1)}(x) - u_n^{(0)}(x) \right|. \quad \ldots\ldots \quad (23)$$

Since $\gamma < 1$, $\sum_{m=0}^{\infty} \gamma^m$ is convergent. Moreover, since $\sum_{n} | u_n^{(0)}(x) | \leqslant c$ on account of (11), we find for all $x \geqslant 0$

$$\sum_{n} \left| u_n^{(1)}(x) - u_n^{(0)}(x) \right| < a \sum_{r} (bcd)^r \sum_{s} (bce)^s$$

$$< a \frac{bcd}{1 - bcd} \cdot \frac{bce}{1 - bce}, \quad \ldots\ldots \quad (24)$$

since both $bcd < 1$ and $bce < 1$.

From (23) we deduce therefore that the doubly infinite series (15) is uniformly convergent for all $x \geqslant 0$.

From this we conclude that all the limit functions

$$u_n(x) = \lim_{m \to \infty} u_n^{(m)}(x), \; (n = 1, 2, 3, \ldots), \quad \ldots\ldots \quad (25)$$

exist and are continuous functions of x.

Moreover for all x we obtain

$$\sum_{n} | u_n(x) | = \sum_{n} \lim_{m \to \infty} | u_n^{(m)}(x) | < 2c. \quad \ldots\ldots \quad (26)$$

From (7) we get, on passing to the limit $m \to \infty$,

$$u_n(x) = f_n(x) + \int_0^x g_n(x,y) \sum_{r,s} \sum_{\substack{k_1, \ldots, k_r \\ l_1, \ldots, l_s}} h_n$$

$$\times u_{k_1}(y) \ldots u_{k_r}(y) u_{l_1}(y) \ldots u_{l_s}(y)\, dy, \qquad \ldots\ldots \quad (27)$$

showing that the function $u_n(x)$ obtained from (25) is a solution of the integral equation (5).

We have to show now the uniqueness of the solution, under the condition that $\sum_n |u_n(x)|$ should be uniformly convergent.

If possible suppose that the system (5) has another solution $v_n(x)$, $(n = 1, 2, \ldots)$ such that

$$\sum_n |v_n(x)| < 2c. \qquad \ldots\ldots \quad (28)$$

Then

$$v_n(x) = f_n(x) + \int_0^x g_n(x,y) \sum_{r,s} \sum_{\substack{k_1, \ldots, k_r \\ l_1, \ldots, l_s}} h_n$$

$$\times v_{k_1}(y) \ldots v_{k_r}(y) v_{l_1}(y) \ldots v_{l_s}(y)\, dy. \qquad \ldots\ldots \quad (29)$$

Then we get from (7) and (29)

$$v_n(x) - u_n^{(m)}(x) = \int_0^x g_n(x,y) \sum_{r,s} \sum_{\substack{k_1, \ldots, k_s \\ l_1, \ldots, l_s}} h_n$$

$$\{v_{k_1}(y) \ldots v_{l_s}(y) - u_{k_1}^{(m-1)}(y) \ldots\ldots u_{l_s}^{(m-1)}(y)\}\, dy$$

$$= \int_0^x g_n(x,y) \sum_{r,s} \sum_{\substack{k_1, \ldots, k_s \\ l_1, \ldots, l_r}} h_n$$

$$\times \{v_{k_1}(y) \ldots\ldots v_{l_{s-1}} [v_{l_s}(y) - u_{l_s}^{(m-1)}(y)]$$
$$+ u_{l_s}^{(m-1)} v_{k_1}(y) \ldots\ldots v_{l_{s-2}}(y) [v_{l_{s-1}}(y) - u_{l_{s-1}}^{(m-1)}(y)]$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ u_{k_2}^{(m-1)}(y)\, u_{k_3}^{(m-1)}(y) \ldots\ldots u_{l_s}^{(m-1)}(y)$$
$$[v_{k_1}(y) - u_{k_1}^{(m-1)}(y)]\}\, dy. \quad \ldots\ldots \quad (30)$$

On account of (14) and (28) we obtain therefore

$$\sum_n | v_n(x) - u_n^{(m)}(x) | < a \sum_{r,s} (r + s)\, b^{r+s}\, q_1(r)\, q_2(s)$$
$$(2c)^{r+s-1} \times \operatorname{Max} \sum_n | v(x) - u_n^{(m-1)}(x) |$$
$$< \frac{2ab^2ce_1}{(1 - 2bcd_1)(1 - 2bce_1)} \cdot \operatorname{Max} \sum_n | v_n(x) - u_n^{(m-1)}(x) |,$$

as found in (19). Repeating this inequality m times, we obtain for all $x \geqslant 0$

$$\sum_n | v_n(x) - u_n^{(m)}(x) | < \gamma^m.$$
$$\times \operatorname{Max} \sum_n | v_n(x) - u_n^{(0)}(x) |. \quad \ldots\ldots \quad (31)$$

where γ is defined in (21). But

$$v_n(x) - u_n^{(0)}(x) = \int_0^x g_n(x, y) \sum_{r,s} \sum_{\substack{k_1, \ldots, k_r \\ l_1, \ldots, l_s}} h_n$$
$$\times v_{k_1}(y) \ldots\ldots v_{k_r}(y)\, v_{l_1}(y) \ldots\ldots v_{l_s}(y)\, dy,$$

so that we get on account of (28) for all $x \geqslant 0$.

$$\sum_n | v_n(x) - u_n^{(0)}(x) | < a \sum_{r,s} b^{r+s}\, q_1(r)\, q_2(s) (2c)^{r+s}$$
$$< a \frac{2bcd}{1 - 2bcd} \frac{2bce}{1 - 2bce}, \quad \ldots\ldots \quad (32)$$

since $q_1(r) < d^r$ and $q_2(s) < e^s$

Therefore, since $\gamma < 1$, we get from (31)

$$\lim_{m\to\infty} \Sigma \mid v_n(x) - u_n^{(m)}(x) \mid = 0. \qquad \ldots\ldots \quad (33)$$

Consequently, for all $x \geqslant 0$

$$v_n(x) = \lim_{m\to\infty} u_n^{(m)}(x) = u_n(x), \qquad \ldots\ldots \quad (34)$$

$(n = 1, 2, 3, \ldots . \infty)$,
showing that the two solutions are identical.

1.3. Solution of the first boundary problem.

Let $p(x)$ be a function > 1 in the interval $0 \leqslant x \leqslant \pi$, and let $p(x)$ and dp/dx be continuous and uniformly bounded in the whole interval.

We consider the non-linear differential equation

$$\frac{\partial}{\partial x}\left\{p(x)\frac{\partial u}{\partial x}\right\} - \frac{\partial u}{\partial t} = P\left(x, t; u, \frac{\partial u}{\partial x}\right), \;\ldots\ldots \quad (1)$$

in the domain

$$0 \leqslant x \leqslant \pi,\; 0 \leqslant t < \infty. \qquad \ldots\ldots\ldots \quad (2)$$

We shall prove that there is one and only one regular solution of (1) which satisfies the conditions

$$u(0, t) = u(\pi, t) = 0 \text{ for all } t \text{ in (2)}, \;\ldots\ldots\ldots \quad (3)$$

and

$$u(x, 0) = f(x) \text{ for all } x \text{ in (2)}. \qquad \ldots\ldots\ldots \quad (4)$$

We assume that the given boundary function $f(x)$ can be expanded in a series of sturm-Liouville eigenfunctions, i.e., the eigenfunctions $\varphi_n(x)$ of the equation

$$\left.\begin{aligned} &\frac{d}{dx}\left\{p(x)\frac{d\varphi}{dx}\right\} + \lambda\varphi = 0, \\ &\varphi(0) = \varphi(\pi) = 0. \end{aligned}\right\} \quad \ldots\ldots \quad (5)$$

Thus

$$f(x) = \sum_n c_n \varphi_n(x), \quad \ldots\ldots\ldots \quad (6)$$

where the coefficients c_n are such that the series $\sum_n \lambda_n^{3/2} c_n$ is absolutely convergent, λ_n being the eigenvalue of (5) corresponding to φ_n. Hence

$$\sum_n \lambda_n^{3/2} \left| c_n \right| = c, \quad \ldots\ldots\ldots \quad (7)$$

where c is an absolute constant.

The question of the expansion of a function in a series of the eigenfunctions $\varphi_n(x)$ has been considered by Lichtenstein.*

The function $P\left(x, t; u, \frac{\partial u}{\partial x}\right)$ is supposed to be analytic in the variables u, $\frac{\partial u}{\partial x}$, so that it can be

* L. Lichtenstein : "Zur Analysis der unendich vielen Variabeln". *Palermo Rendiconti* 38 (1914), 113-166.

expanded in a power series

$$P\left(x, t; u, \frac{\partial u}{\partial x}\right) = \sum_{r,s} p_{r,s}(x, t)\, u^r \left(\frac{\partial u}{\partial x}\right) \quad \ldots. \quad (8)$$

The coefficients $p_{r,s}(x, t)$ and their first two derivatives $\frac{\partial p_{r,s}}{\partial x}$ and $\frac{d^2 p_{r,s}}{\partial x^2}$ are continuous and uniformly bounded in the domain (2). For values of $|u| < 1$, $\left|\frac{\partial u}{\partial x}\right| < 1$, the series (8) will then converge absolutely and uniformly in the whole domain (2).

The solution of the differential equation (1) will be determined as a series of the cigenfunctions $\varphi_n(x)$:

$$u(x, t) = \sum_n v_n(t)\, \varphi_n(x). \quad \ldots\ldots\ldots \quad (9)$$

The conditions (4) and (6) require that the coefficient $v_n(t)$ should have the initial value c_n:

$$v_n(0) = c_n (n = 1, 2, \ldots). \quad \ldots\ldots\ldots \quad (10)$$

Now since $\varphi_n(x)$ is a complete set of orthonormal functions, we have

$$p_{r,s}(x, t)\, u^r \left(\frac{\partial u}{\partial x}\right)^s = \sum_n z_n(r, s; t)\, \varphi_n(x), \quad (11)$$

where

$$z_n(r, s; t) = \sum_{\substack{k_1, \ldots, k_r \\ l_1, \ldots, l_s}} a_n(t, k_1, \ldots, k_r, l_1, \ldots, l_s, r, s)$$

$$\times\, v_{k_1}(t) \ldots v_{k_r}(t)\, v_{l_1}(t) \ldots v_{l_s}(t), \quad \ldots\ldots \quad (12)$$

and $\quad a_n \equiv a_n(t, k_1, \ldots, k_r, l_1, \ldots, l_s, r, s)$

$$= \int_0^\pi p_{r.s}(x, t)\varphi_{k_1}(x)\ldots\varphi_{k_r}(x)\varphi'_{l_1}(x)\ldots\varphi'_{l_s}(x)\varphi_n(x)\,dx, \qquad \ldots\ldots\ldots \quad (13)$$

dashes denoting derivatives with respect to x.

Assuming for the present, what will be proved in the sequal, that the two series

$$\sum_n \lambda_n v_n(t)\varphi_n(x) \text{ and } \sum_n \frac{dv_n}{dt}\varphi_n(x) \qquad \ldots\ldots \quad (14)$$

are absolutely and uniformly convergent in the domain (2), we get on substituting (9) and (11) in (1):

$$-\sum_n \lambda_n v_n(t)\varphi_n(x) - \sum_n \frac{dv_n}{dt}\varphi_n(x) = \sum_n \sum_{r,s} \zeta_n(r, s; t)\varphi_n(x). \qquad \ldots\ldots \quad (15)$$

For each $n \geqslant 1$ we get therefore

$$\frac{dv_n}{dt} + \lambda_n v_n(t) = -\sum_{r,s} \zeta_n(r, s; t), \qquad \ldots\ldots \quad (16)$$

of which the solution satisfying the initial condition (10), viz. $v_n(0) = c_n$, is

$$v_n(t) = c_n e^{-\lambda_n t} - \int_0^t e^{-\lambda_n(t-y)} \sum_{r,s} \zeta_n(r, s; y)\,dy$$

$$= c_n e^{-\lambda_n t} \int_0^t e^{-\lambda_n(t-y)} \sum_{r,s} \sum_{\substack{k_1, \ldots, k_r \\ l_1, \ldots, l_s}} a_n v_{k_1}(y) \ldots$$

$$v_{k_r}(y) v_{l_1}(y) \ldots v_{l_s}(y)\,dy, \qquad \ldots \quad (17)$$

$n = 1, 2, \ldots \infty$.

This is an infinite system of non-linear integral equations for the determination of the coefficients $v_n(t)$ in the solution (9).

We write for all $n \geqslant 1$ and all $t \geqslant 0$:

$$u_n(t) = \lambda_n^{3/2} v_n(t), \qquad \dots\dots\dots \quad (18)$$

$$f_n(t) = \lambda_n^{3/2} c_n e^{-\lambda_n t}, \qquad \dots\dots\dots \quad (19)$$

$$g_n(t, y) = -\lambda_n e^{-\lambda_n (t-y)}, \qquad \dots\dots\dots \quad (20)$$

$$h_n \equiv h_n(y, k_1, \dots, k_r, l_1, \dots, l_s, r, s)$$

$$= \frac{\sqrt{\lambda_n}\, a_n}{(\lambda_{k_1} \dots \lambda_{k_r} \lambda_{l_1} \dots \lambda_{l_s})^{3/2}}. \qquad \dots\dots \quad (21)$$

Then multiplying the equation (17) by $\lambda_n^{3/2}$, we obtain:

$$u_n(t) = f_n(t) + \int_0^t g_n(t, y) \sum_{r, s} \sum_{\substack{k_1, \dots, k_r \\ l_1, \dots, l_s}} h_n u_{k_1}(y) \dots$$

$$u_{k_r}(y)\, u_{l_1}(y) \dots u_{l_s}(y)\, dy, \qquad \dots\dots \quad (22)$$

$(n = 1, 2, \dots \infty)$.

This is the same system of integral equations as considered in § 1·2, except that we have now written t for x.

In order to be able to use the results of that section, we must prove that the functions f_n, g_n, h_n as defined in (19), (20) and (21) satisfy the relations (2), (3) and (4) of § 1·2.

Now, for all $t \geqslant 0$ we have, on account of (7):

$$\sum_n |f_n(t)| = \sum_n |\lambda_n^{3/2} c_n e^{-\lambda_n t}| \leqslant \sum_n |\lambda_n^{3/2} c_n| \leqslant c. \qquad (23)$$

Further

$$\int_0^t g_n(t, y)\, dy = -\lambda_n \int_0^t e^{\lambda_n (t-y)}\, dy = -\lambda_n e^{-\lambda_n t} \int_0^t e^{\lambda_n y}\, dy$$

$$= -\lambda_n e^{-\lambda_n t} \frac{e^{\lambda_n t} - 1}{\lambda_n} = \frac{e^{\lambda_n t} - 1}{e^{\lambda_n t}},$$

so that for all $n \geqslant 1$ and all $t \geqslant 0$, we have

$$\left| \int_0^t g_n(t, y)\, dy \right| < 1. \qquad \ldots\ldots \quad (24)$$

Also the function h_n satisfies the relation (4) § 1.2, as shown in § 1·1.

Therefore, as proved in § 1·2, the sytem of non-linear integral equations (22) has one and only one solution $u_n(t)$, given for all t by

$$u_n(t) = \lim_{m \to \infty} u_n^{(m)}(t), \ (n = 1, 2, \ldots \infty), \qquad \ldots \quad (25)$$

where $u_n^{(m)}(t)$ is the m^{th} successive approximation defined by

$$u_n^{(0)}(t) = f_n(t),$$

and for $m \geqslant 1$

$$u_n^{(m)}(t) = f_n(t) + \int_0^t g_n(t, y) \sum_{r,s} \sum_{\substack{k_1, \ldots, k_r \\ l_1, \ldots, l_s}} h_n$$

$$u_{k_1}^{(m-1)}(y) \ldots u_{k_r}^{(m-1)}(y)\, u_{l_1}^{(m-1)}(y) \ldots u_{l_s}^{(m-1)}(y)\, dy. \qquad (26)$$

It was further proved that $\sum_n |\dot{u}_n(t)|$ is uniformly convergent for all $t \geqslant 0$.

If we now take the coefficients $v_n(t)$ given by

$$v_n(t) = \frac{1}{\lambda_n^{3/2}} u_n(t), \; (n = 1, 2, \ldots \infty), \quad \ldots\ldots \quad (27)$$

then $v_n(t)$ satisfies the integral equation (17) and consequently the differential equation (16). Further, on substituting the value of $v_n(t)$ given by (27) in $\sum_n \sum_{r,s} |z_n(r, s; t)|$, this series is easily verified to be uniformly convergent for all t, on account of the fact that $\sum_n |u_n(t)| = \sum_n \lambda_n^{3/2} |v_n(t)|$ has the same property. Also, since the functions $\varphi_n(x)$ are uniformly bounded for all x in (2) and for all $n \geqslant 1$, the series $\sum_n \lambda_n v_n(t) \varphi_n(x)$ and $\sum_n \sum_{r,s} z_n(r, s; t) \varphi_n(x)$, and consequently $\sum_n \frac{dv_n}{dt} \varphi_n(x)$ are absolutely and uniformly convergent in the domain (2).

Thus

$$u(x, t) = \sum_n v_n(t) \varphi_n(x) \quad \ldots\ldots \quad (28)$$

is the one and only solution of the differential equation (1) satisfying the boundary conditions (3) and (4).

1.4. An Equation of the higher Order.

In § 1.3 we considered an equation of the second order. In this section we shall extend the proof to an equation of 4th, 6th, or any higher even order.

For convenience, we take the case when $p(x) \equiv 1$, so that the sturm-Liouville eigenfunctions $\varphi_n(x)$ degenerate into the sine functions $\sin nx$.

Let m be any positive integer $\geqslant 2$, and let the equation

$$\frac{\partial^{2m} u}{\partial x^{2m}} + (-1)^m \frac{\partial u}{\partial t} = P\left(x, t; u, \frac{\partial u}{\partial x}\right) \qquad \ldots\ldots \quad (1)$$

be considered in the domain $0 \leqslant x \leqslant \pi$, $0 \leqslant t < \infty$ and let it be required to determine a regular solution, satisfying the boundary conditions:

$$u(0, t) = u(\pi, t) = 0 \text{ for all } t; \quad \ldots\ldots \quad (2)$$

$$u(x, 0) = f(x) \text{ for all } x \text{ in } 0 \leqslant x \leqslant \pi. \quad (3)$$

The function $f(x)$ can be expanded in a series

$$f(x) = \sum_n c_n \sin nx,$$

such that $\sum\limits_n n^{2m} |c_n|$ is convergent*: $\qquad \ldots\ldots \quad (4)$

* For this it is sufficient that $\dfrac{d^{2m} f}{dx^{2m}}$ exists, and can be represented as an indefinite integral, cf. E. W. Hobson: *Theory of function of a real variable*, vol. 2, p. 516.

$$\sum_n n^{2m} \, | c_n | = c. \qquad \ldots\ldots \quad (5)$$

The function P satisfies the same conditions as in § 1.3.

Thus $$P\left(x, t; u, \frac{\partial u}{\partial x}\right) = \sum_{r,s} p_{r,s}(x, t) u^r \left(\frac{\partial u}{\partial x}\right)^s \quad (6)$$

The solution of the equation (1) is then determined as

$$u(x, t) = \sum_n v_n(t) \sin nx, \qquad \ldots\ldots\ldots \quad (7)$$

so that on account of (3) and (4) we get

$$v_n(0) = c_n \quad (n = 1, 2, \ldots). \qquad \ldots\ldots \quad (8)$$

Then

$$p_{r,s}(x, t) u^r \left(\frac{\partial u}{\partial x}\right)^s = \sum_n z(r, s; t) \sin nx, \quad (9)$$

where

$$z_n(r, s; t) = \sum_{\substack{k_1, \ldots, k_r \\ l_1, \ldots, l_s}} a_n v_{k_1}(t) \ldots v_{k_r}(t) v_{l_1}(t) \ldots v_{l_s}(t),$$

$$a_n \equiv a_n(t, k_1, \ldots, k_r, l_1, \ldots, l_s, r, s,)$$
$$= \int_0^\pi p_{r,s}(x, t) \sin k_1 x \ldots \sin k_r x \cos l_1 x \ldots$$
$$\cos l_s x \sin nx \, dx. \qquad \ldots \qquad \ldots \quad (11)$$

Substituting (6), (7) and (9) in (1), we obtain

$$(-1)^m \sum_n n^{2m} v_n(t) \sin nx + (-1)^m \sum_n \frac{dv_n}{dt} \sin nx$$
$$= \sum_n \sum_{r,s} z_n(r, s; t) \sin nx, \qquad \ldots\ldots \quad (12)$$

where all the series in (12) have been assumed to be absolutely and uniformly convergent. This will be proved later.

From (12) we get for each $n \geqslant 1$:

$$\frac{dv_n}{dt} + n^{2m} v_n(t) = (-1)^m \sum_{r,s} z_n(r, s; t), \quad \dots \quad (13)$$

of which the solution satisfying (8) is

$$v_n(t) = c_n e^{-n^{2m} t} + (-1)^m \int_0^t e^{-n^{2m}(t-y)} \sum_{r,s} z_n(r, s; y)\, dy$$

$$= c_n e^{-n^{2m} t} + (-1)^m \int_0^t e^{-n^{2m}(t-y)} \sum_{r,s} \sum_{\substack{k_1, \dots, k_r \\ l_1, \dots, l_s}} a_n$$

$$\times v_{k_1}(y) \dots v_{k_r}(y)\, v_{l_1}(y) \dots v_{l_s}(y)\, dy. \quad (14)$$

We set for all $t \geqslant 0$ and all $n \geqslant 1$:

$$u_n(t) = n^{2m} v_n(t); \quad \dots\dots \quad (15)$$

$$f_n(t) = n^{2m} c_n e^{-n^{2m} t}; \quad \dots\dots \quad (16)$$

$$g_n(t, y) = (-1)^m n^{2m} e^{-n^{2m}(t-y)}; \quad \dots\dots \quad (17)$$

$$h_n \equiv h_n(y; k_1, \dots, k_r, l_1, \dots, l_s; r, s)$$

$$= \frac{a_n}{(k_1 \dots k_r l_1 \dots l_s)^{2m}} \quad \dots\dots \quad (18)$$

From (14) we obtain, on multiplying both sides by n^{2m},

$$u_n(t) = f_n(t) + \int_0^t g_n(t, y) \sum_{r,s} \sum_{\substack{k_1, \dots, k_r \\ l_1, \dots, l_s}} h_n$$

$$u_{k_1}(y) \dots u_{k_r}(y)\, u_{l_1}(y) \dots u_{l_s}(y)\, dy, \quad \dots\dots \quad (19)$$

$(n = 1, 2, \dots \infty)$.

It can be proved as before that f_n', g_n, h_n satisfy inequalities (2) (3) and (4) of § 1.2. Hence the infinite system of non-linear integral equations can be solved uniquely by the method of successive approximations. From $u_n(t)$ we can then determine $v_n(t)$ according to the relations

$$v_n(t) = n^{-2m} u_n(t) \quad (n = 1, 2, \ldots). \quad \ldots\ldots \quad (20)$$

Since $\sum_n | u_n(t) |$ is uniformly convergent for all t, it can be easily verified that all the series in (12) are absolutely and uniformly convergent.

Hence

$$u(x, t) = \sum_n v_n(t) \sin nx$$

is the required solution of (1) satisfying the boundary conditions (2) and (3), and this solution is unique.

1.5. An Elliptic-Parabolic Equation.

We consider the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = u^2, \quad \ldots\ldots \quad (1)$$

and determine a solution $u(x, y\,;\,t)$ which is regular in the domain R defined by

$$0 \leqslant x \leqslant \pi, \quad 0 \leqslant y \leqslant \pi, \quad 0 \leqslant t < \infty, \quad \ldots\ldots \quad (2)$$

and which satisfies the boundary conditions:

$$\left.\begin{array}{l} u(0, y;\ t) = u(\pi, y;\ t) = 0 \text{ for all } y \text{ and } t \text{ in R;} \\ u(x, 0;\ t) = u(x, \pi;\ t) = 0 \text{ for all } x \text{ and } t \text{ in R;} \end{array}\right\} \quad (3)$$

and

$$u(x, y;\ 0) = f(x, y) \text{ for all } x \text{ and } y \text{ in R.} \quad \ldots\ldots \quad (4)$$

We assume that $f(x, y)$ can be expanded in the double Fourier series

$$f(x, y) = \sum_{m,n} c_{m,n} \sin mx \sin ny, \quad \ldots\ldots \quad (5)$$

and that the series $\sum_{m,n} (m^2 + n^2) \mid c_{m,n} \mid$ is convergent.

For the solution of (1) we write

$$u(x, y;\ t) = \sum_{m,n} v_{m,n}(t) \sin mx \sin ny, \quad \ldots\ldots \quad (6)$$

and get

$$u^2(x, y;\ t) = \sum_{m,n} z_{m,n}(t) \sin mx \sin ny, \quad \ldots\ldots \quad (7)$$

where

$$z_{m,n}(t) = \sum_{\varkappa\lambda\mu\nu} f_m(\varkappa, \lambda) f_n(\mu, \nu) v_{\varkappa,\mu}(t) v_{\lambda,\nu}(t), \quad (8)$$

with

$$\left.\begin{array}{l} f_m(\varkappa, \nu) = \dfrac{2}{\pi} \displaystyle\int_0^\pi \sin \varkappa x \sin \lambda x \sin mx \, dx, \\ f_n(\mu, \nu) = \dfrac{2}{\pi} \displaystyle\int_0^\pi \sin \mu y \sin \nu y \sin ny \, dy. \end{array}\right\} \quad (9)$$

From (1), (6) and (7) we get then

$$- \sum (m^2 + n^2) v_{m,n}(t) \sin mx \sin ny$$

$$-\sum \frac{dv_{m,n}}{dt} \sin mx \sin ny$$

$$= \sum z_{m,n}(t) \sin mx \sin ny; \qquad \ldots\ldots \quad (10)$$

therefore for all $m, n \geqslant 1$

$$\frac{dv_{m,n}}{dt} + (m^2 + n^2)\, v_{m,n}(t) = -\, z_{m,n}(t). \quad \ldots\ldots \quad (11)$$

We have assumed that all the series in (10) are absolutely and uniformly convergent in the domain R. As usual, this can be proved later.

From (4) and (5) we see that the solution (6) would have to satisfy the initial condition

$$v_{m,n}(0) = c_{m,n} \quad (m, n = 1, 2, \ldots). \quad \ldots\ldots \quad (12)$$

The solution of (11) which satisfies (12) is given by:

$$v_{m,n}(t) = c_{m,n}\, e^{-(m^2+n^2)t} - \int_0^t e^{-(m^2+n^2)(t-s)}\, z_{m,n}(s)\, ds$$

$$= c_{m,n}\, e^{-(m^2+n^2)t} - \int_0^t e^{-(m^2+n^2)(t-s)} \sum_{\kappa\lambda\mu\nu} f^m(\kappa, \lambda)$$

$$f_n(\mu, \nu)\, v_{\kappa,\mu}(s)\, v_{\lambda,\nu}(s)\, ds. \quad \ldots\ldots \quad (13)$$

We write for all $m, n \geqslant 1$:

$$w_{m,n}(t) = (m^2 + n^2)\, v_{m,n}(t), \quad c'_{m,n} = (m^2 + n^2)\, c_{m,n}, \quad \ldots\ldots \quad (14)$$

and get from (13)

$$w_{m,n}(t) = c'_{m,n} e^{-(m^2+n^2)t} - (m^2 + n^2)\int_0^t e^{-(m^2+n^2)(t-s)}$$

$$\times \sum_{\kappa\lambda\mu\nu} \frac{f_m(\varkappa, \lambda) f_n(\mu, \nu)}{(\varkappa^2 + \mu^2)(\lambda^2 + \nu^2)} w_{\kappa,\mu}(s) w_{\lambda,\nu}(s)\, ds \quad \ldots \quad (15)$$

This is a doubly infinite system of non-linear integral equations for the determination of the functions $w_{m,n}(t)$ which in their turn determine the Fourier coefficients $v_{m,n}(t)$. We shall solve this system by the method of successive approximations, and for this purpose we set

$$w^{(0)}_{m,n}(t) = c'_{m,n} e^{-(m^2+n^2)t}, \qquad \ldots\ldots \quad (16)$$

and for all $r \geqslant 1$

$$w^{(r)}_{m,n}(t) = c'_{m,n} e^{-(m^2+n^2)t} - (m^2 + n^2)\int_0^t e^{-(m^2+n^2)(t-s)}$$

$$\times \sum_{\kappa\lambda\mu\nu} \frac{f_m(\varkappa, \lambda) f_n(\mu, \nu)}{(\varkappa^2 + \mu^2)(\lambda^2 + \nu^2)} w^{(r-1)}_{\kappa,\mu}(s)\, w^{(r-1)}_{\lambda,\nu}(s)\, ds. \quad (17)$$

We have to prove the convergence of this approximation. We shall first show that the doubly infinite series

$$\sum_{m,n} \frac{|f_m(\varkappa, \lambda) f_n(\mu, \nu)|}{(\varkappa^2 + \mu^2)(\lambda^2 + \nu^2)} \qquad \ldots\ldots \quad (18)$$

is uniformly convergent for all $\varkappa\lambda\mu\nu$.

From (9) we have, on writing $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$:

$$f_m(\varkappa, \lambda) f_n(\mu, \nu) = -\frac{4}{(m^2+n^2)\pi^2}\int_0^\pi\int_0^\pi (\sin \varkappa x$$

$$\sin \mu y \sin \lambda x \sin \nu y) \times \Delta (\sin mx \sin ny)\, dx\, dy$$

$$= -\frac{4}{(m^2+n^2)\pi^2}\int_0^\pi\int_0^\pi (\sin mx \sin ny)$$

$$\times \Delta (\sin \varkappa x \sin \mu y \sin \lambda x \sin \nu y)\, dxdy$$

$$= -\frac{4}{(m^2+n^2)\pi^2}\int_0^\pi\int_0^\pi \sin mx \sin ny$$

$$\{-(\varkappa^2+\lambda^2)\sin \varkappa x \sin \mu y \times \sin \lambda x \sin \nu y$$

$$+ 2\varkappa\lambda \cos \varkappa x \cos \lambda x \sin \mu y \sin \nu y - (\mu^2+\nu^2) \sin$$

$$\varkappa x \sin \mu y \sin \lambda x \sin \nu y + 2\mu\nu \sin \varkappa x \sin \lambda x \cos \mu y$$

$$\cos \nu y\}\, dx\, dy.$$

Therefore

$$\frac{|f_m(\varkappa, \lambda) f_n(\mu, \nu)|}{(\varkappa^2+\mu^2)(\lambda^2+\nu^2)} < \frac{4}{(m^2+n^2)}\Big\{|f_m(\varkappa, \lambda) f_n(\mu, \nu)|$$

$$+ g_m(\varkappa, \lambda) f_n(\mu, \nu) + |f_m(\varkappa, \lambda) g_n(\mu, \nu)|\Big\}, \quad (19)$$

where

$$\left.\begin{aligned} g_m(\varkappa, \lambda) &= \frac{2}{\pi}\int_0^\pi \cos \varkappa x \cos \lambda x \sin mx\, dx, \\ g_n(\mu, \nu) &= \frac{2}{\pi}\int_0^\pi \cos \mu x \cos \nu x \sin nx\, dx. \end{aligned}\right\} \quad (20)$$

From Parseval's theorem we know that for all $\varkappa, \lambda, \mu, \nu$:

$$\sum_n f_m^2(\varkappa, \lambda) \leqslant 2, \quad \sum_n f_n^2(\mu, \nu) \leqslant 2,$$
$$\sum_n g_m^2(\varkappa, \lambda) \leqslant 2, \quad \sum_n g_m^2(\mu, \nu) \leqslant 2.$$

Therefore

$$\sum_{m,n} f_m^2(\varkappa, \lambda) f_n^2(\mu, \nu) \leqslant 4, \quad \sum_{m,n} g_n^2(\varkappa, \lambda) f_m^2(\mu, \nu) \leqslant 4,$$
$$\sum_{m,n} f_m^2(\varkappa, \lambda) g_n^2(\mu, \nu) \leqslant 4.$$

We get from (19) on summing, squaring and using Schwaz's inequality as well as the inequality $2ab \leqslant a^2 + b^2$:

$$\left\{\sum_{m,n} \frac{| f_m(\varkappa, \lambda) f_n(\mu, \nu) |}{(\varkappa^2 + \mu^2)(\lambda^2 + \nu^2)}\right\}^2 \leqslant \left\{\sum_{m,n} \frac{16}{(m^2 + n^2)^2}\right\}$$
$$\sum_{m,n} \Big\{ | f_m(\varkappa, \lambda) f_n(\mu, \nu) | + | g_m(\varkappa, \lambda) f_n(\mu, \nu) |$$
$$+ | f_m(\varkappa, \lambda) g_n(\mu, \nu) | \Big\}^2$$
$$\leqslant \sum_{m,n} \frac{16 \times 3}{(m^2 + n^2)^2} \cdot \sum_{m,n} \Big\{ f_m^2(\varkappa, \lambda) f_n^2(\mu, \nu)$$
$$+ g_n^2(\varkappa, \lambda) f_n^2(\mu, \nu) + f_m^2(\varkappa, \lambda) g_n^2(\mu, \nu) \Big\}$$
$$< \sum_{m,n} \frac{16 \times 3 \times 12}{(m^2 + n^2)^2}.$$

But the doubly infinite series $\sum_{m,n} \frac{1}{(m^2 + n^2)^2}$ is

convergent, so that

$$\left\{\sum_{m,n} \frac{|f_m(\varkappa, \lambda) f_m^2(\mu, \nu)|}{(\varkappa^2 + \mu^2)(\lambda^2 + \nu^2)}\right\}^2$$

and with it the series (18) is uniformly convergent for all $\varkappa, \lambda, \mu, \nu$:

$$\sum_{m,n} \frac{|f_m(\varkappa, \lambda) f_n(\mu, \nu)|}{(\varkappa^2 + \mu^2)(\lambda^2 + \nu^2)} = a, \quad \ldots\ldots \quad (21)$$

where a is an absolute constant.

We write, since $\sum\limits_{m,n} (m^2 + n^2)\, c_{m,n}$ is assumed convergent,

$$\sum_{m,n} |c'_{m,n}| = \sum_{m,n} (m^2 + n^2)\, |c_{m,n}| \doteq c. \qquad (22)$$

We have also for all $t > 0$ and all $m, n \geqslant 1$:

$$\left|\int_0^t e^{-(m^2+n^2)(t-s)}\, ds\right| = \left|e^{-(m^2+n^2)t} \int_0^t e^{(m^2+n^2)s}\, ds\right|$$

$$= \left|\frac{1}{e^{(m^2+n^2)t}} \frac{e^{(m^2+n^2)t} - 1}{(m^2 + n^2)}\right| < \frac{1}{(m^2 + n^2)}. \qquad (23)$$

Thus from (21) and (23) we get for all $t > 0$ and all $\varkappa, \lambda, \mu, \nu \geqslant 1$:

$$\sum_{m,n} (m^2 + n^2) \int_0^t e^{-(m^2+n^2)(t-s)}\, ds . \frac{|f_m(\varkappa, \lambda) f_n(\mu, \nu)|}{(\varkappa^2 + \mu^2)(\lambda^2 + \nu^2)}$$

$$< \sum_{m,n} (m^2 + n^2) \cdot \frac{1}{(m^2 + n^2)} \cdot \frac{|f_m(\varkappa, \lambda) f_n(\mu, \nu)|}{(\varkappa^2 + \mu^2)(\lambda^2 + \nu^2)}$$

$$< a. \quad \ldots\ldots\ldots\ldots \quad (24)$$

Then we can show as in § 1.2, that the approximations (17) are convergent, and that the system of non-linear integral equations (13) has one and only one solution. The only alteration in the procedure would be to change the simple summation over n to a double summation over m and n. Thus it is seen that the non-linear elliptic-parabolic equation (1) has a unique solution satisfying the boundary conditions (3) and (4).

1.6. The mixed boundary problem.

In the previous sections we considered only the first boundary problem in whch u had given values at both extremes of the x-interval. The method is, however, capable of dealing with other kinds of boundary problems. We shall consider here the third boundary problem, the so called mixed problem, in which the value of u is given at one end of the x-interval, and that of $\frac{\partial u}{\partial x}$ is given at the other end.

Thus we try to determine a solution $u(x, t)$ of

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = \sum_r p_r(x, t)\, u^r, \qquad \ldots\ldots \quad (1)$$

in the domain

$$0 \leqslant x \leqslant \pi, \quad 0 \leqslant t < \infty, \qquad \ldots\ldots \quad (2)$$

such that $u(x, t)$ satisfies the mixed conditions

$$u(0, t) = 0, \frac{\partial u}{dx}(\pi, t) = 0 \text{ for all } t \geqslant 0, \quad \ldots\ldots \quad (3)$$

and

$$u(x, 0) = f(x) \text{ for all } x \text{ in } 0 \leqslant x \leqslant \pi. \quad \ldots\ldots \quad (4)$$

This problem can be brought to depend upon the boundary problem we have hitherto considered by continuing the functions $u(x, t)$, $p_r(x, t)$ and $f(x)$ in the interval $\pi \leqslant x \leqslant 2\pi$ according to the method of images. Thus we take for all $t \geqslant 0$:

$$\left.\begin{aligned} u(2\pi - x, t) &= u(x, t), \\ p_r(2\pi - x, t) &= p_r(x, t), \\ f(2\pi - x) &= f(x), \end{aligned}\right\} \quad \ldots\ldots \quad (5)$$

where x varies in the interval $0 \leqslant x \leqslant \pi$.

Now we define functions $u_1(x, t)$, $p_r'(x, t)$ and $f_1(x)$ in the interval $0 \leqslant x \leqslant 2\pi$, for all $t \geqslant 0$ as follows:

$$\left.\begin{aligned} u_1(x, t) &= u(x, t) \text{ in } 0 \leqslant x \leqslant \pi, \\ &= u(2\pi - x, t) \text{ in } \pi \leqslant x \leqslant 2\pi; \end{aligned}\right\} \quad \ldots\ldots \quad (6)$$

$$\left.\begin{aligned} p_r'(x, t) &= p_r(x, t) \text{ in } 0 \leqslant x \leqslant \pi, \\ &= p_r(2\pi - x, t) \text{ in } \pi \leqslant x \leqslant 2\pi; \end{aligned}\right\} \quad \ldots\ldots \quad (7)$$

$$\left.\begin{aligned} f_1(x) &= f(x) \text{ in } 0 \leqslant x \leqslant \pi, \\ &= f(2\pi - x) \text{ in } \pi \leqslant x \leqslant 2\pi. \end{aligned}\right\} \quad \ldots\ldots \quad (8)$$

Then we see that $u_1(x, t)$ is a regular solution of

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = \sum_r p_r'(x, t)\, u^r, \qquad \ldots\ldots \quad (9)$$

in the domain

$$0 \leqslant x \leqslant 2\pi, \quad 0 \leqslant t < \infty, \qquad \ldots\ldots \quad (10)$$

which satisfies the boundary conditions:

$$\left.\begin{aligned} u_1(0, t) &= u(0, t) = 0, \\ u_1(2\pi, t) &= u(2\pi - 2\pi, t) = u(0, t) = 0, \end{aligned}\right\} \qquad \ldots\ldots \quad (11)$$

and

$$u_1(x, 0) = u(x, 0) = f(x) \text{ in } 0 \leqslant x \leqslant \pi,$$
$$u_1(x, 0) = u(2\pi - x, 0) = f(2\pi - x) \text{ in } \pi \leqslant x \leqslant 2\pi,$$

so that

$$u_1(x, 0) = f_1(x) \text{ in } \quad 0 \leqslant x \leqslant 2\pi. \qquad \ldots\ldots \quad (12)$$

But the method of § 1.3 is applicable to an interval of any length, and therefore the solution $u_1(x, t)$ of the equation (9) is unique.

We define another function $u_2(x, t)$ in the domain (10) by the relation

$$u_2(x, t) = u_1(2\pi - x, t). \qquad \ldots\ldots \quad (13)$$

Then we get for all x and t in the domain (10):

$$\frac{\partial^2 u_2(x, t)}{\partial x^2} - \frac{\partial u_2(x, t)}{\partial t} = \frac{\partial^2 u_1(2\pi - x, t)}{\partial x^2} - \frac{\partial u_1(2\pi - x, t)}{\partial t}$$

$$= \sum_r p'_r(2\pi - x, t)u_1^r(2\pi - x, t)$$

$$= \sum_r p'_r(x, t)u_2^r(x, t), \quad \ldots \quad (14)$$

showing that $u_2(x, t)$ satisfies the differential equation (9). Further

$$\left.\begin{aligned} u_2(0, t) &= u_1(2\pi, t) = 0 \\ u_2(2\pi, t) &= u_1(0, t) = 0 \,; \end{aligned}\right\} \quad \ldots\ldots \quad (15)$$

and for all x in $0 \leqslant x \leqslant 2\pi$

$$u_2(x, 0) = u_1(2\pi - x, 0) = f_1(2\pi - x) = f_1(x). \quad \ldots\ldots \quad (16)$$

Equations (15) and (16) signify that $u_2(x, t)$ satisfies the same boundary conditions as $u_1(x, t)$. It must therefore be identical with $u_1(x, t)$:

$$u_2(x, t) \equiv u_1(x, t),$$

i.e., on account of (13),

$$u_1(2\pi - x, t) \equiv u_1(x, t). \quad \ldots\ldots \quad (17)$$

Hence

$$\frac{\partial u_1}{\partial x}(\pi, t) = 0. \quad \ldots\ldots \quad (18)$$

Thus, in the domain (2) the function $u_1(x, t)$ satisfies the differential equation (1) and the boundary conditions (3) and (4). It is therefore the required solution.

1.7. Non-rectangular Domain.

So far we have considered only rectangular domains

$$0 \leqslant x \leqslant \pi, \quad 0 \leqslant t \leqslant T, \quad (T \to \infty). \quad \ldots\ldots \quad (1)$$

Now we take a general domain D bounded by the x-axis between the points 0 and π, and the two curves C_1 and C_2 with the equations

$$(C_1)\ x = h_1(t), \quad (C_2)\ x = h_2(t), \qquad \ldots\ldots \quad (2)$$

where

$$h_1(0) = 0, \quad h_2(0) = \pi, \quad h_2(t) > h_1(t) \text{ for all } t. \quad (3)$$

The problem is to determine the solution $u(x, t)$ of the differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = \sum_r p_r(x, t)\, u^r, \qquad \ldots\ldots \quad (4)$$

which is regular in D, and which satisfies the boundary conditions :

$$\left.\begin{array}{l} u = 0 \text{ on } C_1 \text{ and on } C_2, \\ u = f(x) \text{ for } t = 0. \end{array}\right\} \qquad \ldots\ldots \quad (5)$$

We shall prove that on transforming to new variables, the problem can be reduced to the determination of the solution of

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = \sum_r q_r(x, t)\, u^r, \qquad \ldots\ldots \quad (6)$$

in a rectangular domain of the type (1), and for analogous boundary conditions. Our integration method as developed in the previous sections would then be applicable to it.

For this purpose we first make the transformation

$$x = h_1(t) + x' \, g(t)/\pi, \; t = t, \qquad \ldots\ldots \quad (7)$$

where

$$g(t) = h_2(t) - h_1(t). \qquad \ldots\ldots \quad (8)$$

To the domain D of the (x, t) plane would then correspond the rectangular domain

$$0 \leqslant x' \leqslant \pi, \quad 0 \leqslant t, \qquad \ldots\ldots \quad (9)$$

of the (x', t) plane. The differential equation (4) is then transformed to the equation

$$\frac{\partial^2 u}{\partial x'^2} - \frac{1}{\pi^2} g^2(t) \frac{\partial u}{\partial t} = \frac{1}{\pi^2} g^2(t) \sum_r p_r(x, t) \, u^r \qquad \ldots \quad (10)$$

If we transform further

$$t' = \pi^2 \int_0^t \frac{dt}{g^2(t)}, \qquad \ldots\ldots \quad (11)$$

and write

$$\left.\begin{aligned} u'(x', t') &= u(x, t), g'(t') = g(t), \\ p_r'(x', t') &= p_r(x, t), \; f'(x') = f(x), \end{aligned}\right\} \qquad \ldots\ldots \quad (12)$$

then we see that u- satisfies the differential equation

$$\frac{\partial^2 u'}{\partial x'^2} - \frac{\partial u'}{\partial t'} = \frac{1}{\pi^2}\left\{g'(t')\right\}^2 \sum_r p'_r(x', t')\, u'^r, \qquad (13)$$

and the boundary conditions become:

$$\left.\begin{aligned} &u'(0, t') = u'(\pi, t') = 0 \text{ for all } t' \geqslant 0, \\ &u'(x', 0) = f'(x') \text{ for all } x' \text{ in } 0 \leqslant x' \leqslant \pi. \end{aligned}\right\} \qquad (14)$$

If we write

$$q'_r(x', t') = \frac{1}{\pi^2}\left\{g'(t')\right\}^2 p'_r(x', t'), \quad \ldots \quad (15)$$

and drop suffixes throughout, the equation (13) is seen to be the same as (6). Further, it is obvious from (11) that $t' \to \infty$ with t.

The proposed reduction has thus been obtained.

CHAPTER II

NON-LINEAR HYPERBOLIC EQUATIONS

2·1. Solution of a system of non-linear integral equations in the restricted domain.

We consider now an infinite system of non-linear integral equations

$$w_n(x) = f_n(x) + \int_0^x g_n(x,y) \sum_{k_1, k_2, \ldots, k_r} h_n \\ w_{k_1}(y)\, w_{k_2}(y) \ldots w_{k_r}(y)\, dy, \qquad \ldots\ldots \quad (1)$$

$(n = 1, 2, \ldots \infty)$, where $f_n(x)$, $g_n(x,y)$ and $h_n(y, k_1, \ldots, k_r, r)$ are sequences defined for the values of x, y in the domain

$$0 \leqslant x \leqslant T, \quad 0 \leqslant y \leqslant x \leqslant T, \qquad \ldots\ldots \quad (2)$$

and satisfying the relations

$$\sum_n |f_n(x)| \leqslant c, \text{ for all } x \text{ in } (2); \qquad \ldots\ldots \quad (3)$$

$$|g_n(x,y)| \leqslant B, \text{ for all } x, y \text{ in } (2) \text{ and all } n \geqslant 1; \qquad \ldots\ldots \quad (4)$$

$$\sum_n |h_n| \leqslant A\, b^r\, p(r), \text{ for all } y \text{ in } (1) \text{ and all } k_1, \ldots, k_r \geqslant 1. \qquad \ldots\ldots \quad (5)$$

A, B, b, c are again absolute constants, and $p(r)$ denotes a polynomial in r. It will be seen that the functions f_n and h_n satisfy inequalities similar to those in § 1·2, while $g_n(x,y)$ itself is uniformly bounded now, whereas its integral was assumed to be uniformly bounded in § 1·2. This fact brings about the fundamental difference that whereas the solution of the system (5) § 1·2 existed for all values of x however large, the solution of the system (1) will exist only for a restricted domain x.

We transform the equation (1) by writing

$$u_n(x) = w_n(x) - f_n(x), \quad (n = 1, 2, \ldots), \qquad \ldots\ldots \quad (6)$$

so that (1) becomes

$$u_n(x) = \int_0^x g_n(x,y) \sum_{k_1, \ldots, k_r} h_n(f_{k_1}(y) + u_{k_1}(y)) \ldots$$
$$(f_{k_r}(y) + u_{k_r}(y))\, dy, \qquad \ldots\ldots \quad (7)$$
$$(n = 1, 2, \ldots \infty).$$

We use again the method of successive approximations to solve the system (7), and write

$$u_n^{(0)}(x) = 0, \text{ and for all } m \geqslant 1:$$

$$u_n^{(m)}(x) = \int_0^x g_n(x,y) \sum_{k_1, \ldots, k_r} h_n(f_{k_1}(y) + u_{k_1}^{(m-1)}(y)) \ldots$$
$$(f_{k_r}(y) + u_{k_r}^{(m-1)}(y))\, dy. \qquad \ldots\ldots \quad (8)$$

We have to show now that these approximations are convergent. For this purpose we prove first that the series $\sum_n |u_n^{(m)}(x)|$ is uniformly convergent for all x in a certain interval, and for all $m \geqslant 1$.

Let

$$v_m = \operatorname{Max} \sum_n |u_n^{(m)}(x)| \text{ for all } x \text{ in } (2) \quad \ldots\ldots \quad (9)$$
$$(m = 0, 1, 2, \ldots \infty).$$

Then from (8) we find on account of (3), (4) and (5):

$$v_m \leqslant ABT\, b^r\, p(r)\, (c + v_{m-1})^r. \quad \ldots\ldots \quad (10)$$

We define another sequence $\{R_m\}$ as follows:

$$R_0 = 0,$$
$$R_m = 2^{r-1}\, a\, T\, b^r\, p(r)\, (c^r + R_{m-1}^r), \ldots\ldots \quad (11)$$
$$(m = 1, 2, \ldots \infty),$$

where we have written $AB = a$. Now if α and β are any positive real numbers, and if r is any integer, we have the inequality $(\alpha + \beta)^r \leqslant 2^{r-1} (\alpha^r + \beta^r)$. Thus from (10) and (11) we see that for all $m \geqslant 0$:

$$v_m \leqslant R_m, \quad \ldots\ldots \quad (12)$$

so that the sequence v_m is certainly bounded if the sequence R_m is bounded.

Let q be some fixed positive integer, and let for all $m \leqslant q$

$$R_m \leqslant 1. \qquad \ldots\ldots \quad (13)$$

Then from (11) it follows that for $m \leqslant q$,

$$\begin{aligned} R_{m+1} - R_m &= 2^{r-1} a\, T\, b^r p(r)\ (R_m^r - R_{m-1}^r) \\ &= l_r T\, (R_m^{r-1} + R_m^{r-2} R_{m-1} + R_{m-1}^{r-3} R_{m-1}^2 + \ldots \\ &\qquad + R_{m-1}^{r-1})\,(R_m - R_{m-1}), \end{aligned} \qquad \ldots\ldots \quad (14)$$

where we have written

$$l_r = 2^{r-1} a\, b^r p(r). \qquad \ldots\ldots \quad (15)$$

From (14) we get for $m \leqslant q$:

$$|\,R_{m+1} - R_m\,| \leqslant r\, l_r\, T\, |\,R_m - R_{m-1}\,|. \quad \ldots \quad (16)$$

But since

$$\begin{aligned} R_{q+1} = R_1 + (R_2 - R_1) + (R_3 - R_2) + \ldots \\ + (R_{q+1} - R_q), \end{aligned}$$

we have from (16):

$$\begin{aligned} |\,R_{q+1}\,| &\leqslant l_r\, T c^r \{1 + (r l_r T) + (r l_r T)^2 + \ldots (r l_r T)^q\} \\ &< \frac{l_r\, T c^r}{1 - r l_r T}, \end{aligned} \qquad \ldots\ldots \quad (17)$$

provided $r l_r T < 1$.

Now if we select T so that

$$T < \mathrm{Min}\left\{\frac{1}{2 r l_r},\ \frac{1}{2 l_r c^r}\right\}, \qquad \ldots \quad (18)$$

Then

$$R_{q+1} < 2l_r Tc^r < 1. \qquad \ldots\ldots \quad (19)$$

We find therefore that if $R_q < 1$ then $R_{q+1} < 1$. Since $R_1 = l_r Tc^r < 1$, it follows by induction that

$$R_m < 1, \quad (m = 0, 1, 2, \ldots \infty). \qquad \ldots\ldots \quad (20)$$

From (9) and (11) we conclude that

$$\sum_n | u_n^{(m)}(x) | < 1, \qquad \ldots\ldots \quad (21)$$

for all $m \geqslant 1$, and all x which lie in the restricted domain given by (18).

We go on to prove now that the double series

$$\sum_{m=0}^{\infty} \sum_n | u_n^{(m+1)}(x) - u_n^{(m)}(x) | \qquad \ldots\ldots \quad (22)$$

is also uniformly convergent for all x in $0 \leqslant x \leqslant T$.

From (8) we have

$$\begin{aligned} u_n^{(m+1)}(x) - u_n^{(m)}(x) \\ = \int_0^x g_n(x,y) \sum_{k_1, \ldots, k_r} h_n \{ (f_{k_1} + u_{k_1}^{(m)}) \ldots (f_{k_r} + u_{k_r}^{(m)}) \\ - (f_{k_1} + u_{k_1}^{(m-1)}) \ldots (f_{k_r} + u_{k_r}^{(m-1)}) \} \, dy \\ = \int_0^x g_n(x,y) \sum_{k_1, \ldots, k_r} h_n [\{ \sum_{j=1}^{r} f_{k1} f_{k2} \cdots f_{k_{j-1}} f_{kj+1} \cdots \\ f_{k_r} (u_{k_j}^{(m)} - u_{k_j}^{(m-1)}) \} \\ + \sum_{\binom{r}{2}} f_{k1} \cdots f_{k_r-2} \{ u_{k_{r-1}}^{(m)} u_{k_r}^{(m)} - u_{k_{r-1}}^{(m-1)} u_{k_r}^{(m-1)} \} \\ + \sum_{\binom{r}{3}} f_{k1} \cdots f_{k_r-3} \{ u_{k_{r-2}}^{(m)} u_{k_{r-1}}^{(m)} u_{k_r}^{(m)} - u_{k_{r-2}}^{(m-1)} u_{k_{r-1}}^{(m-1)} u_{k_r}^{(m-1)} \} \end{aligned}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \sum_{\binom{r}{r-1}} f_{k_1} \{u_{k_2}^{(m)} u_{k_3}^{(m)} \dots u_{k_r}^{(m)} - u_{k_2}^{(m-1)} u_{k_3}^{(m-1)} \dots u_{k_r}^{(m-1)}\}$$

$$+ \{u_{k_1}^{(m)} u_{k_2}^{(m)} \dots u_{k_r}^{(m)} - u_{k_1}^{(m-1)} u_{k_2}^{(m-1)} \dots u_{k_r}^{(m-1)}\}]dy. \quad \dots\dots \quad (23)$$

Using the inequalities that $\sum_n |f_n(x)| \leqslant c$ and $\sum_n |u_n^{(m)}(x)| < 1$ for and all x in (2) all $m \geqslant 1$, we get from (23):

$$\sum_n |u_n^{(m+1)}(x) - u_n^{(m)}(x)| < BAb^r p(r)T \Big\{ \binom{r}{1} c^{r-1} + 2\binom{r}{2} c^{r-2} + 3\binom{r}{3} c^{r-3} + \dots + (r-1)\binom{r}{r-1} c + r \Big\} \operatorname{Max} \sum_n |u_n^{(m)}(x) - u_n^{(m-1)}(x)|,$$

$$< \alpha_{r,c}\, T \operatorname{Max} \sum_n |u_n^{(m)}(x) - u_n^{(m-1)}(x)|, \quad (24)$$

where

$$\alpha_{r,c} = a\, b^r p(r) \Big\{ \binom{r}{1} c^{r-1} + 2\binom{r}{2} c^{r-2} + \dots + r \Big\} \quad (25)$$

We see therefore that the series $\sum_n |u_n^{(m+1)}(x) - u_n^{(m)}(x)|$ is uniformly convergent if the series $\sum_n |u_n^{(m)}(x) - u_n^{(m-1)}(x)|$ has the same property. Repeating the inequality (25) m times we obtain

$$\sum_n |u_n^{(m+1)}(x) - u_n^{(m)}(x)| < (\alpha_{r,c}\, T)^m \operatorname{Max.} \sum_n |u_n^{(1)}(x) - u_n^{(0)}(x)| < (\alpha_{r,c}\, T)^m. \quad \dots\dots \quad (26)$$

We take the domain T so small that

$$T \leqslant \frac{\gamma}{\alpha_{r,c}}, \qquad \ldots\ldots \quad (27)$$

where γ is any number < 1. Then we get from (26):

$$\sum_{m=0}^{\infty} \sum_{n} | u_n^{(m+1)}(x) - u_n^{(m)}(x) | < \sum_{m=0}^{\infty} \gamma^m < \frac{1}{1-\gamma}, \quad (28)$$

showing that the double series (22) is uniformly convergent, as required.

It follows that all the limit-functions

$$u_n(x) = \lim_{m \to \infty} u_n^{(m)}(x), \quad (n = 1, 2, \ldots) \qquad (29)$$

exist and are continuous. Further, we get on account of (21) and (29):

$$\sum_{n} | u_n(x) | < 1. \qquad \ldots\ldots \quad (30)$$

from (8) we get, on passing to the limit $m \to \infty$,

$$u_n(x) = \int_0^x g_n(x, y) \sum_{k_1, \ldots, k_r} h_n\big(f(y) + u_{k_1}(y)\big) \ldots$$
$$\big(f_{k_r}(y) + u_{k_r}(y)\big)\, dy. \qquad \ldots\ldots \quad (31)$$

$(n = 1, 2, \ldots \infty)$.

Thus $u_n(x)$, as found from (29), satisfies the integral equation (7), therefore $w_n(x) = u_n(x) + f_n(x)$ will satisfy the equation (1).

Thus the existence of a solution $w_n(x)$ of (1) is established for sufficiently small T. We shall now prove that this is the only solution such that $\sum\limits_n |w_n(x)|$ is uniformly convergent. Obviously, it is sufficient for this purpose to show that the system (7) has no other solution $\bar{u}_n(x)$ $(n = 1, 2, \ldots)$, different from $u_n(x)$ as found in (29), which is such that $\sum\limits_n |\bar{u}_n(x)|$ converges uniformly.

If possible, let $\bar{u}_n(x)$ be another solution of (7), so that

$$\bar{u}_n(x) = \int_0^x g_n(x, y) \sum_{k_1, \ldots, k_r} h_n\big(f_{k_1}(y) + \bar{u}_{k_1}(y)\big) \cdots$$
$$\big(f_{k_r}(y) + \bar{u}_{k_r}(y)\big)\, dy. \quad \ldots\ldots \quad (32)$$

Also let $$\sum_n |\bar{u}_n(x)| < \bar{c}, \quad \ldots\ldots \quad (33)$$

for all x in $0 \leqslant x \leqslant T$, $\bar{c}$ being a constant.

Then from (32) and (8) we find

$$\bar{u}_n(x) - u_n^{(m)}(x) = \int_0^x g_n(x, y) \sum_{k_1, \ldots, k_r} h_n \Big\{ \big(f_{k_1} + \bar{u}_{k_1}\big)$$
$$\big(f_{k_2} + \bar{u}_{k_2}\big) \cdots \big(f_{k_r} + \bar{u}_{k_r}\big) - \big(f_{k_1} + u_{k_1}^{(m-1)}\big)$$
$$\big(f_{k_2} + u_{k_2}^{(m-1)}\big) \cdots \big(f_{k_r} + u_{k_r}^{(m-1)}\big) \Big\}\, dy. \quad (34)$$

From (34) we find as before, for all x in $0 \leqslant x \leqslant T$,

$$\sum_n |\bar{u}_n(x) - u_n^{(m)}(x)| < \beta(r, c, \bar{c})T. \operatorname{Max} \sum_n |\bar{u}_n(x) - u_n^{(m-1)}(x)|, \quad \ldots\ldots \quad (35)$$

where

$$\beta(r, c, \bar{c}) = a\, b^r p(r) \Big\{ \binom{r}{1} c^{r-1} + \binom{r}{2} c^{r-2} (\bar{c} + 1) + \binom{r}{3} c^{r-3} (\bar{c}^{-2} + \bar{c} + 1) + \ldots + \binom{r}{3-1} c(\bar{c}^{r-1} + \bar{c}^{r-2} + \ldots + \bar{c} + 1) + (\bar{c}^r + \bar{c}^{r-1} + \ldots + \bar{c} + 1) \Big\} \quad \ldots\ldots \quad (36)$$

Repeating the inequality (35) m times, we obtain

$$\sum_n |\bar{u}_n(x) - u_n^{(m)}(x)| < \{\beta(r, c, \bar{c})T\}^m \times \operatorname{Max} \sum_n |\bar{u}_n(x) - u_n^{(0)}(x)| < \bar{c}\, \{\beta(r, c, c)T\}^m. \quad (37)$$

We assume T to be so small that

$$T \leqslant \frac{\gamma'}{\beta(r, c, \bar{c})}, \quad \ldots\ldots \quad 38$$

where γ' is any number < 1. Then

$$\lim_{m \to \infty} \sum_n |\bar{u}_n(x) - u_n^{(m)}(x)| \leqslant \bar{c} \lim_{m \to \infty} \gamma'^m = 0 \quad \ldots\ldots \quad (39)$$

Hence we find that for all x in $0 \leqslant x \leqslant T$:

$$\bar{u}_n(x) = \lim_{m \to \infty} u_n^{(m)}(x) = u_n(x), \quad \ldots\ldots \quad (40)$$

($n = 1, 2, \dots \infty$), provided only that T satisfies the inequalities (18), (27) and (38). For this restricted domain the solution of (1) is therefore unique.

2·2. Solution of the non-linear hyperbolic equation.

In this section we consider the non-linear hyperbolic equation

$$\frac{\partial}{\partial x}\left\{p(x)\frac{\partial u}{\partial x}\right\} - \frac{\partial^2 u}{\partial t^2} = p_r(x, t)\, u^r \qquad \dots\dots \quad (1)$$

in the domain

$$0 \leqslant x \leqslant \pi, \quad 0 \leqslant t \leqslant T. \qquad \dots\dots \quad (2)$$

$p(x)$ is > 1, and $p(x)$ and dp/dx are continuous and uniformly bounded in the whole interval. r is any given positive integer, while $p_r(x, t)$, $\dfrac{\partial p_r}{\partial x}$, $\dfrac{\partial^2 p_r}{\partial x^2}$ are continuous and uniformly bounded in the domain (2).

We wish to determine a solution of (1) which is regular in the domain (2), and which satisfies the boundary conditions :

$$u(0, t) = u(\pi, t) = 0 \text{ for all } t \text{ in (2)}; \qquad \dots\dots \quad (3)$$

$$u(x, 0) = f_1(x), \ \frac{\partial u}{\partial t}(x, 0) = f_2(x) \text{ for all } x \text{ in (2)}. \quad (4)$$

The functions $f_1(x)$, $f_2(x)$ can be expanded in a series of Sturm-Liouville eigenfunctions $\varphi_n(x)$:

$$f_1(x) = \sum_n c_n \varphi_n(x), \quad f_2(x) = \sum_n d_n \varphi_n(x), \qquad \ldots\ldots \quad (5)$$

where we assume that the series

$$\sum_n \lambda_n \,|\, c_n \,| \text{ and } \sum_n \sqrt{\lambda_n} \,|\, d_n \,| \qquad \ldots\ldots \quad (6)$$

are convergent.

For the solution of (1) we write as usual.

$$u(x, t) = \sum_n w_n(t)\, \varphi_n(x). \qquad \ldots\ldots \quad (7)$$

The conditions (4) will be satisfied if the coefficients $w_n(t)$ satisfy the relations

$$w_n(0) = c_n, \frac{dw_n}{dt}(0) = d_n, \qquad \ldots\ldots \quad (8)$$

$$(n = 1, 2, \ldots \infty).$$

Further we have

$$p_r(x, t)\, u^r = \sum_n z_n(t)\, \varphi_n(x), \qquad \ldots\ldots \quad (9)$$

where

$$z_n(t) = \int_0^\pi p_r(x, t)\, u^r(x, t)\, \varphi_n(x)\, dx$$

$$= \sum_{k_1, \ldots, k_r} a_n(t, k_1, \ldots, k_r)\, w_{k_1}(t)\, w_{k_2}(t) \ldots w_{k_r}(t),$$

$$\ldots\ldots \quad (10)$$

with

$$a_n(t, k_1, \ldots, k_r) = \int_0^\pi p_r(x, t)\, \varphi_{k_1}(x) \ldots$$
$$\varphi_{k_r}(x)\, \varphi_n(x)\, dx. \quad \ldots\ldots \quad (11)$$

We assume, what we shall prove later, that the series

$$\sum_n \lambda_n w_n(t)\, \varphi_n(x) \text{ and } \sum_n \frac{d^2 w_n}{dt^2} \varphi_n(x) \quad \ldots \quad (12)$$

are absolutely and uniformly convergent in the domain (2). Then substituting (7) and (9) in (1) we obtain

$$-\sum_n \lambda_n w_n(t)\, \varphi_n(x) - \sum_n \frac{d^2 w_n}{dt^2} \varphi_n(x)$$
$$= \sum_n z_n(t)\, \varphi_n(x), \quad \ldots\ldots \quad (13)$$

so that for all $n \geqslant 1$ we have

$$\frac{d^2 w_n}{dt^2} + \lambda_n w_n(t) = -z_n(t). \quad \ldots\ldots \quad (14)$$

The solution of this equation which satisfies the initial conditions (8) is given by

$$w_n(t) = c_n \cos \sqrt{\lambda_n}\, t + \frac{d_n}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}\, t$$
$$- \frac{1}{\sqrt{\lambda_n}} \int_0^t \sin \sqrt{\lambda_n}\,(t - y)\, z_n(y)\, dy$$

$$= c_n \cos \sqrt{\lambda_n}\, t + \frac{d_n}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}\, t$$

$$- \frac{1}{\sqrt{\lambda_n}} \int_0^t \sin \sqrt{\lambda_n}\,(t-y)$$

$$\times \sum_{k_1, \dots, k_r} a_n w_{k_1}(y) \dots w_{k_r}(y)\, dy, \quad \dots\dots \quad (15)$$

$$(n = 1, 2, \dots \infty).$$

This is an infinite system of non-linear integral equations for determining the coefficients $w_n(t)$. We transform the system by writing

$$w_n(t) = c_n \cos \sqrt{\lambda_n}\, t + \frac{d_n}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}\, t + \frac{1}{\lambda_n} u_n(t), \quad \dots\dots \quad (16)$$

$$f_n(t) = \lambda_n c_n \cos \sqrt{\lambda_n}\, t + \sqrt{\lambda_n}\, d_n \sin \sqrt{\lambda_n}\, t. \quad (17)$$

Then we get

$$u_n(t) = -\sqrt{\lambda_n} \int_0^t \sin \sqrt{\lambda_n}\,(t-y) \sum_{k_1, \dots, k_r} \frac{a_n}{\lambda_{k_1} \dots \lambda_{k_r}}$$

$$\{f_{k_1}(y) + u_{k_1}(y)\} \dots \{f_{k_r}(y) + u_{k_r}(y)\}\, dy, \quad \dots \quad (18)$$

$$(n = 1, 2, \dots \infty).$$

We write for all $n \geqslant 1$:

$$g_n(t, y) = -\sin \sqrt{\lambda_n}\,(t-y), \quad \dots\dots \quad (19)$$

$$h_n(t, k_1, \dots, k_r) = \frac{\sqrt{\lambda_n}\, a_n(t, k_1, \dots, k_r)}{\lambda_{k_1} \lambda_{k_2} \dots \lambda_{k_r}}. \quad \dots\dots \quad (20)$$

Then (18) becomes

$$= \int_0^t g_n(x,y) \sum_{k_1, \ldots, k_r} h_n \{f_{k_1}(y) + u_{k_1}(y)\} \ldots \{f_{k_r}(y) + u_{k_r}(y)\} dy, \qquad \ldots\ldots \quad (21)$$

$(n = 1, 2, \ldots \infty)$.

From (6) and (17) we get for all t in (2),

$$\sum_n | f_n(t) | \leqslant c \,; \ (c \text{ absolute constant}). \quad \ldots\ldots \quad (22)$$

Also for all $n \geqslant 1$ and all t in (2) we have

$$| g_n(t,y) | = | -\sin \lambda_n (t-y) | \leqslant 1. \quad \ldots\ldots \quad (23)$$

Further, from (8) § 1.1 we obtain (since $s = 0$):

$$\sum_n | h_n | < A\, b^r p(r), \text{ for all } t \text{ in } (2), \quad \ldots\ldots \quad (24)$$

and all $k_1, \ldots, k_r \geqslant 1$.

From § 2.1 we see therefore that one and only one solution of the system (21) exists for a restricted interval of t, and also that $\sum_n | u_n(t) |$ is uniformly convergent in this interval.

Having found $u_n(t)$, we get the coefficients $w_n(t)$ from the relation (16). When these values of $w_n(t)$ are substituted in $z_n(t)$ as given by (10), we find that the series $\sum_n | z_n(t) |$ is uniformly convergent for t in $0 \leqslant t \leqslant T$. Further, we

find from (16), on account of (17) and (22), th $\sum_n \lambda_n \mid w_n(t) \mid$ is also uniformly convergent. functions $\varphi_n(x)$ are uniformly bounded for al x in $0 \leqslant x \leqslant \pi$ and for all $n \geqslant 1$. Therefore $\sum_n \lambda_n w_n(t) \varphi_n(x)$ and $\sum_n \zeta_n(t) \varphi_n(x)$ are absolutely and uniformly convergent in the domain (2); consequently from (13) we see that $\sum_n \frac{d^2 w_n}{dt^2} \varphi_n(x)$ is also absolutely and uniformly convergent in (2).

Thus

$$u(x, t) = \sum_n w_n(t) \varphi_n(x) \qquad \ldots\ldots \quad (25)$$

is the unique solution of (1) which satisfies the boundary conditions (3) and (4).

Obviously there is no difficulty in extending the result to the equation

$$\frac{\partial}{\partial x}\left\{ p(x) \frac{\partial u}{\partial x} \right\} - \frac{\partial^2 u}{\partial t^2} = \sum_{r=1}^{j} p_r(x, t)\, u^r, \qquad (26)$$

where j is any finite integer.

2·3. Non-vanishing boundary values.

In the theory developed so far we have assumed the boundary values to be zero. This was not done from any necessity, but only with a view

implicity.* We shall prove now that non-
shing boundary values are also amenable to
treatment.

Let it be required to determine the regular solution of

$$\frac{\partial}{\partial x}\left\{p(x)\frac{\partial u}{\partial x}\right\}-\frac{\partial^2 u}{\partial t^2}=p_r(x, t)\,u^r, \quad \ldots\ldots \quad (1)$$

in the domain

$$0 \leqslant x \leqslant \pi, \quad 0 \leqslant t \leqslant T, \quad \ldots\ldots \quad (2)$$

satisfying the boundary conditions

$$u(0, t)=\psi_1(t),\ u(\pi, t)=\psi_2(t) \text{ for all } t \text{ in } (2), \quad (3)$$

and the initial conditions

$$u(x, 0)=f_1(x),\ \partial u(x, 0)/\partial t=f_2(x) \text{ for all } x \text{ in } (2). \quad (4)$$

Let $f_1(x)$ and $f_2(x)$ be capable of expansion in Fourier series

$$f_1(x)=\sum_n c_n'\,\varphi_n(x), \quad f_2(x)=\sum_n c_n''\varphi_n(x), \quad (5)$$

such that the series

$$\sum_n \lambda_n\,|\,c_n'\,| \text{ and } \sum_n \sqrt{\lambda_n}\,|\,c_n''\,|$$

are convergent. ...... (6)

In order that the conditions (3) and (4) are compatible, we must have

$$\psi_1(0)=\frac{d\psi_1(0)}{dt}=\frac{d^2\psi_1(0)}{dt^2}=0, \quad \ldots\ldots \quad (7)$$

$$\psi_2(0) = \frac{d\psi_2(0)}{dt} = \frac{d^2\psi_2(0)}{dt^2} = 0.$$

Let $g(x)$ be any function defined in $0 \leqslant x \leqslant \frac{\pi}{3}$ such that $g, \frac{dg}{dx}, \frac{d^2g}{dx^2}, \frac{d^3g}{dx^3}, \frac{d^4g}{dx^4}$ are continuous, and

$$\left.\begin{aligned} &g(0) = \frac{dg}{dx}(0) = \frac{d^2g}{dx^2}(0) = \frac{d^3g}{dx^3}(0) = \frac{d^4g}{dx^4}(0) = 0 \\ &g\left(\frac{\pi}{3}\right) = 1, \frac{dg}{dx}\left(\frac{\pi}{3}\right) = \frac{d^2g}{dx^2}\left(\frac{\pi}{3}\right) = \frac{d^3g}{dx^3}\left(\frac{\pi}{3}\right) \\ &\qquad = \frac{d^4g}{dx^4}\left(\frac{\pi}{3}\right) = 0. \end{aligned}\right\} \quad (8)$$

Then we define a function $F(x, t)$ as follows:

$$F(x,t) = \begin{cases} F_1(x, t) = \psi_1(t) + \dfrac{x^2}{2}\left\{\dfrac{d^2\psi_1(t)}{dt^2} + \left(\dfrac{d\psi_1}{dt}\right)^2\right\} \\ \qquad\qquad \text{for } 0 \leqslant x \leqslant \dfrac{\pi}{3}, \\ F_2(x, t) = \psi_2(t) + \dfrac{(x-\pi)^2}{2}\left\{\dfrac{d^2\psi_2}{dt^2} + \left(\dfrac{d\psi_2}{dt}\right)^2\right\} \\ \qquad\qquad \text{for } \dfrac{2\pi}{3} \leqslant x \leqslant \pi, \\ F_1(x, t) + \{F_2(x, t) - F_1(x, t)\}\, g\left(x - \dfrac{\pi}{3}\right) \\ \qquad\qquad \text{for } \dfrac{\pi}{3} \leqslant x \leqslant \dfrac{2\pi}{3}. \end{cases} \quad \ldots\ldots \quad (9)$$

Then we see that F, $\frac{\partial F}{\partial t}$, $\frac{\partial^2 F}{\partial t^2}$ and their first four derivatives w. r. to x are continuous. Also

$$F(0, t) = \psi_1(t), \quad F(\pi, t) = \psi_2(t). \quad \ldots\ldots \quad (10)$$

The functions $F(x, 0)$ and $\frac{\partial F}{\partial t}(x, 0)$ can be expanded in Fourier series

$$F(x, 0) = \sum_n b'_n \varphi_n(x), \quad \frac{\partial F}{\partial t}(x, 0) = \sum_n b''_n \varphi_n(x), \quad (11)$$

with

$$\sum_n \lambda_n \mid b'_n \mid \text{ and } \sum_n \sqrt{\lambda_n} \mid b''_n \mid \text{ convergent.} \quad (12)$$

We define another function $H(x, t)$ by

$$H(x, t) = -\frac{\partial}{\partial x}\left\{p(x)\frac{\partial F}{\partial x}\right\} + \frac{\partial^2 F}{\partial t^2} + p_r(x, t)\, F^r. \quad (13)$$

Then H, $\frac{\partial H}{\partial x}$ and $\frac{\partial^2 H}{\partial x^2}$ are continuous in the domain (2). It can also be verified that

$$H(0, t) = H(\pi, t) = 0. \quad \ldots\ldots \quad (14)$$

Hence $H(x, t)$ can be expanded in a series

$$H(x, t) = \sum_n h_n(t)\, \varphi_n(x), \quad \ldots\ldots \quad (15)$$

where

$$\sum_n \sqrt{\lambda_n} \mid h_n(t) \mid \text{ is uniformly convergent.} \quad (16)$$

Now we consider the function

$$w(x, t) = u(x, t) - F(x, t). \qquad \ldots\ldots \quad (17)$$

It satisfies the differential equation

$$\frac{\partial}{\partial x}\left\{ p(x) \frac{\partial w}{\partial x}\right\} - \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left\{ p(x) \frac{\partial u}{\partial x}\right\} - \frac{\partial^2 u}{\partial t^2}$$
$$- \left[\frac{\partial}{\partial x}\left\{ p(x) \frac{\partial F}{\partial x}\right\} - \frac{\partial^2 F}{\partial t^2}\right]$$
$$= p_r(x, t)\ \{u^r(x, t) - F^r(x, t)\} + H(x, t) \quad (18)$$

We substitute for $u(x, t)$ in the right hand side of (18) by writing $u = w + F$, and get

$$\frac{\partial}{\partial x}\left\{ p(x) \frac{\partial w}{\partial x}\right\} - \frac{\partial^2 w}{\partial t^2}$$
$$= H(x, t) + q_r^{(1)}(x, t)\, w + q_r^{(2)}(x, t)\, w^2 + \ldots$$
$$+ q_r^{(r)}(x, t)\, w^r$$
$$= H(x, t) + \sum_{k=1}^{r} q_r^{(k)}(x, t)\, w^k(x, t), \qquad \ldots\ldots \quad (19)$$

where the functions $q_r^{(k)}(x, t)$ can be easily calculated in terms of r, $p_r(x, t)$ and the powers of $F(x, t)$. For example

$$q_r^{(1)}(x, t) = r\, p_r(x, t)\, F^{r-1}(x, t),$$
$$q_r^{(2)}(x, t) = \frac{r(r-1)}{2}\, p_r(x, t)\, F^{r-2}(x, t), \ldots \quad (20)$$

etc.

The function $w(x, t)$ satisfies the boundary conditions

$$\left.\begin{aligned} w(0, t) &= u(0, t) - F(0, t) \\ &= \psi_1(t) - \psi_1(t) = 0, \\ w(\pi, t) &= u(\pi, t) - F(\pi, t) \\ &= \psi_2(t) - \psi_2(t) = 0\,; \end{aligned}\right\} \quad (21)$$

$$\left.\begin{aligned} w(x, 0) &= u(x, 0) - F(x, 0) \\ &= f_1(x) - F(x, 0) = h_1(x)\ (\text{say}), \\ \frac{\partial w}{\partial t}(x, 0) &= \frac{\partial u}{\partial t}(x, 0) - \frac{dF}{\partial t}(x, 0) \\ &= f_2(x) - \frac{\partial F}{\partial t}(x, 0) = h_2(x)\ (\text{say}). \end{aligned}\right\} \quad (22)$$

We see from (5) and (11) that the boundary functions $h_1(x)$ and $h_2(x)$ can be expanded in series

$$h_1(x) = \sum_n h'_n \varphi_n(x), \quad h_2(x) = \sum_n h''_n \varphi_n(x), \quad \ldots \quad (23)$$

and that on account of (6) and (12) the series

$$\sum_n \lambda_n \,|\, h'_n \,| \text{ and } \sum_n \sqrt{\lambda_n} \,|\, h''_n \text{ are convergent.} \quad \ldots \quad (24)$$

As shown in (26) § 2.2, the equation (19) for $w(x, t)$ can be solved for the boundary values (21) and (22), and then the solution $u(x, t)$ of the equation (1), satisfying the boundary conditions (3) and (4) can be obtained by taking

$$u(x, t) = w(x, t) + F(x, t). \quad \ldots\ldots \quad (25)$$

CHAPTER III

NON-LINEAR ELLIPTIC EQUATIONS

3·1. Resumé of the results for linear elliptic equations

3·1 (1). The boundary problem for $\nabla^2 u = 0$.

Let D be a region bounded by a simple closed contour C, and consider the equation

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad \ldots\ldots \quad (1)$$

It is proved in the potential theory that we can find a solution $u(x, y)$ of (1) which is regular in D and continuous on C, and which is such that the expression

$$\alpha u + \beta \frac{\partial u}{\partial n} \qquad \ldots\ldots \quad (2)$$

takes given values on C. The curve C, the function α, β and the boundary values can have a fairly general character.*

* *c. f.* The articles on potential theory by Lichtenstein in the Enzyklopaedie der Mathematirchen Wissenscheften Bd. II (Analysis) *C*.

3·1 (2). The Green's function.

The Green's function for C is defined as a harmonic function (solution of $\nabla^2 u = 0$) of ξ, η which vanishes on C, and which is continuous in D except at the point (x, y) where it becomes infinite as $-\frac{1}{2}\log\{(\xi - x)^2 + (\eta - y)^2\}$.

Let $g(x, y)$ be the harmonic function regular in D, continuous on C, and taking on C the same values as $\frac{1}{2}\log\{(\xi - x)^2 + (\eta - y)^2\}$. From the previous paragraph we know that such a function exists. The Green's function is therefore

$$G(\xi, \eta\,; x, y) = g(x, y) - \tfrac{1}{2}\log\{(\xi - x)^2 + (\eta - y)^2\}. \qquad \ldots\ldots \quad (1)$$

The function $g(x, y)$ has derivatives $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ perfectly determined in D, and $\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$ are also determined in D except at the point $\xi = x$, $\eta = y$ where they become infinite as $\frac{\xi - x}{(\xi - x)^2 + (\eta - y)^2}$ and $\frac{\eta - y}{(\xi - x)^2 + (\eta - y)^2}$ respectively.

Let

$$A(x, y) = \frac{1}{2\pi}\iint_D G(\xi, \eta\,; x, y)\, d\xi\, d\eta. \qquad \ldots\ldots \quad (2)$$

Then $A(x, y)$ is shown to have a maximum M when (x, y) moves in D, and that

$$M \to 0 \text{ when } D \to 0. \qquad \ldots\ldots \quad (3)$$

Similarly let

$$\left.\begin{aligned} B(x, y) &= \frac{1}{2\pi}\iint_D \frac{\partial G}{\partial x}\, d\xi\, d\eta, \\ C(x, y) &= \frac{1}{2\pi}\iint_D \frac{\partial G}{\partial y}\, d\xi\, d\eta \end{aligned}\right\} \qquad \ldots\ldots \quad (4)$$

Then it is proved that $B(x, y)$ and $C(x, y)$ have maxima such that

$$| B(x, y) | \leqslant N, \quad | C(x, y) | \leqslant N, \qquad \ldots\ldots \quad (5)$$

and also that

$$N \to 0 \text{ when } D \to 0. \qquad \ldots\ldots \quad (6)$$

Thus we see that M and N depend only on the curve C, varying continuously with C and tending to zero with C.

3·1 (3). The boundary problem for $\triangle^2 u = f(x, y)$.

Consider the equation

$$\triangle^2 u = f(x, y), \qquad \ldots\ldots \quad (1)$$

and suppose that we wish to determine a solution which is continuous in D and which vanishes on

C. It is easily seen that the solution is given by the integral

$$u(x, y) = -\frac{1}{2\pi}\iint_D f(\xi, \eta)\, G(\xi, \eta\,;\, x, y)\, d\xi d\eta. \quad (2)$$

Suppose F is the maximum of $f(x, y)$ in D and on C. Then we get for all x, y in D and on C:

$$|\, u(x, y)\, | < FM. \qquad \ldots\ldots \quad (3)$$

Further we have

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{1}{2\pi}\iint_D f(\xi, \eta)\, \frac{\partial G}{\partial x}\, d\xi\, d\eta, \\ \frac{\partial u}{\partial y} &= -\frac{1}{2\pi}\iint_D f(\xi, \eta)\, \frac{\partial G}{\partial y}\, d\xi\, d\eta. \end{aligned}\right\} \quad \ldots \quad (4)$$

Then from (5) § 3·1 (2) we obtain for all x, y in D and on C

$$\left|\frac{\partial u}{\partial x}\right| < FN, \left|\frac{\partial u}{\partial y}\right| < FN. \qquad \ldots\ldots \quad (5)$$

The derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ of (2) exist provided $f(x, y)$ is continuous. But the continuity of $f(x, y)$ is not sufficient for the establishment of the existence of the second derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, and of the relation $\triangle^2 u = f$. This will be the case if f satisfies the relation

$$|\, f(x_1, y_1) - f(x_0, y_0)\, | < K\, r^{\mu}, \qquad \ldots\ldots \quad (6)$$

where (x_1, y_1) and (x_0, y_0) are any two points in D, r the distance between them, and K and μ two constants.

However, if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous in D, then $u(x, y)$ has second derivatives which are themselves continuous in D, and which remain continuous if the point (x, y) approaches indefinitely near to the contour C. Further, if

$$\text{Max} \left|\frac{\partial f}{\partial x}\right| \leqslant F_1, \text{ Max} \left|\frac{\partial f}{\partial y}\right| \leqslant F_1, \qquad \dots \quad (7)$$

then it is found that

$$\text{Max} \left|\frac{\partial^2 u}{\partial x^2}\right| < \lambda F + M_1 F_1, \text{ Max} \left|\frac{\partial^2 u}{\partial y^2}\right| < \lambda F + M_1 F_1, \qquad \dots\dots \quad (8)$$

where λ and M_1 are two positive quantities independent of $f(x, y)$, and where

$$M_1 \to 0 \text{ when } D \to 0. \qquad \dots\dots \quad (9)$$

3·2. The non-linear equation $\Delta^2 u = P\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$.

In this section we wish to determine a solution $u(x, y)$ of the equation

$$\Delta^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = P(x, y, u, p, q) \quad \ldots\ldots \quad (1)$$

$\left[p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y}\right]$, which is regular in the domain D and takes given values on the curve C. Suppose that the function P remains finite and well determined when

$$|u|, |p|, |q| < L \quad \ldots\ldots \quad (2)$$

for all x, y in D, L being a positive constant. Then we see that P will have a certain maximum μ.

We now take the following successive approximations in order to solve the problem.

Take any function $u_1(x, y)$ such that

$$|u_1|, |p_1|, |q_1| < L, \quad \ldots\ldots \quad (3)$$

$$p_n = \frac{\partial u_n}{\partial x}, \; q_n = \frac{\partial u_n}{\partial y} \; (n = 1, 2, 3, \ldots),$$

and write

$$\Delta^2 u_2 = P(x, y, u_1, p_1, q_1) \quad \ldots\ldots \quad (4)$$

$$\Delta^2 u_3 = P(x, y, u_2, p_2, q_2) \quad \ldots\ldots \quad (5)$$

$$\Delta^2 u_n = P(x, y, u_{n-1} \, p_{n-1}, q_{n-1}) \quad \ldots\ldots \quad (6)$$

Since we know the function $u_1(x, y)$, the right hand member of (4) is a known function of x, y, say $f(x,y)$. Thus (4) is of the form $\nabla^2 u_2 = f(x,y)$

which has been considered in § 3·1(3). We can therefore find a solution $u_2(x, y)$ which is regular in D and takes given values on C. Then we can determine $u_3(x,y)$ from the equation (5), and in general we can find $u_n(x,y)$ satisfying the equation (6) and taking given values on C.

To establish the convergence of these approximations we proceed as follows.

Taking M and N to be the quantities defined in § 3·1(2), we know that M and N tend to zero continuously with D. We choose D sufficiently small so that

$$\mu M < L, \mu N < L. \qquad \ldots\ldots \quad (7)$$

Now $P(x,y,u_1,p_1,q_1)$ has the maximum μ, so that on account of (5) and (3) § 3·1(3) we get

$$|u_2| < \mu M < L, \ |p_2| < \mu N < L, \ |q_2| < \mu N < L.$$

But from (2) we know that as soon as u_2, p_2 and q_2 lie between $-L$ and $+L$, the function $P(x,y,u_2,p_2,q_2)$ will have the maximum μ. Then from the equation (5) we conclude that

$$|u_3| < \mu M < L, \ |p_3| < \mu N < L, \ |q_3| < \mu N < L.$$

Repeating this argument we see that in general

$$|u_n| < \mu M < L, \ |p_n| < \mu N < L,$$
$$|q_n| < \mu N < L, \qquad \ldots\ldots \quad (8)$$

$(n = 1, 2, 3, \ldots)$.

We write

$$v_n = u_n - u_{n-1} \; (n = 3, 4, \ldots), \qquad \ldots\ldots \quad (9)$$

and get from (6) and (9)

$$\nabla^2 v_3 = P(x, y, u_2, p_2, q_2) - P(x, y, u_1, p_1, q_1), \qquad \ldots\ldots \quad (10)$$

$$\nabla^2 v_n = P(x, y, u_{n-1}, p_{n-1}, q_{n-1}) - P(x, y, u_{n-2}, p_{n-2}\, q_{n-2}). \qquad \ldots\ldots \quad (11)$$

From (9) we see that all $v_n(x, y)$ vanish on C.

We assume that the function $P(x, y, u, p, q)$ satisfies Lipschitz Condition in u, p, q, so that

$$| P(x, y, u_2, p_2, q_2) - P(x, y, u_1, p_1, q_1) | < A | u_2 - u_1 | + B | p_2 - p_1 + C | q_2 - q_1 |, \qquad (12)$$

where A, B, C are positive constants.

Thus, for all n we obtain from (11)

$$| \nabla^2 v_n | < A | v_{n-1} | + B \left| \frac{\partial v_{n-1}}{\partial x} \right| + C \left| \frac{\partial v_{n-1}}{\partial y} \right|, \qquad \ldots\ldots \quad (13)$$

$(n = 4, 5, \ldots)$.

On account of (8) and (12) we see that the right hand member of (10) has a maximum F such that

$$F < 2L(A + B + C). \qquad \ldots\ldots \quad (14)$$

Hence from (5) and (3) § 3·1 (3) we obtain

$$|v_3| < MF,\ \left|\frac{\partial v_3}{\partial x}\right| < NF,\ \left|\frac{\partial v_3}{\partial y}\right| < NF.$$

Consequently

$$A|v_3| + B\left|\frac{\partial v_3}{\partial x}\right| + C\left|\frac{\partial v_3}{\partial y}\right| < (AM + BN + CN)F$$
$$< RF,$$

where

$$R = AM + BN + CN. \qquad \ldots\ldots \quad (15)$$

From this we get

$$|v_4| < MRF,\ \left|\frac{\partial v_4}{\partial x}\right| < N.RF,\ \left|\frac{\partial v_4}{\partial y}\right| < N.RF,$$

and therefore

$$A|v_4| + B\left|\frac{\partial v_4}{\partial x}\right| + C\left|\frac{\partial v_4}{\partial y}\right| < R.RF = R^2F.$$

Thus in general we find for all $n \geqslant 4$:

$$|v_n| < M.R^{n-3}F,\ \left|\frac{\partial v_n}{\partial x}\right| < N.R^{n-3}F,$$
$$\left|\frac{\partial v_n}{\partial y}\right| < N.R^{n-3}F. \qquad \ldots\ldots \quad (16)$$

From (16) we conclude that v_n, $\frac{\partial v_n}{\partial x}$, $\frac{\partial v_n}{\partial y}$ will tend to zero, and consequently u_n will tend to a limit if

$$R = AM + BN + CN < 1. \qquad \ldots\ldots \quad (17)$$

Now M and N tend to zero with D, so that the Condition (17) will be certainly satisfied if D is taken sufficiently small. In this case the limit function $u(x, y)$ is represented by the infinite series

$$u = u_2 + v_3 + v_4 + \dots + v_n + \dots \qquad \dots\dots \quad (18)$$

This will be a continuous function of x, y, which has the first derivatives given by the series

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u_2}{\partial x} + \frac{\partial v_3}{\partial x} + \dots + \frac{\partial v_n}{\partial x} + \dots \\ \frac{\partial u}{\partial y} &= \frac{\partial u_2}{\partial y} + \frac{\partial v_3}{\partial y} + \dots + \frac{\partial v_n}{\partial y} + \dots \end{aligned}\right\} \qquad (19)$$

The function $u(x, y)$ will take on C the same values as u_2, and therefore the given values.

Further, since

$$u_n(x, y) = -\frac{1}{2\pi}\iint_D P\left(\xi, \eta, u_{n-1}, \frac{\partial u_{n-1}}{\partial \xi}, \frac{\partial u_{n-1}}{\partial \eta}\right) G(\xi, \eta\,; x, y)\, d\xi\, d\eta,$$

where $G(\xi, \eta\,; x, y)$ is the Green's function for C, we get on passing to the limit $n \to \infty$:

$$u(x, y) = -\frac{1}{2\pi}\iint_D P\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) G(\xi, \eta\,; x, y)\, d\xi\, d\eta. \qquad \dots\dots \quad (20)$$

We have seen that $u(x, y)$ has first derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$. We must prove now that the second derivatives also exist, and satisfy the differential equation (1).

We assume that $P(x, y, u, p, q)$ has first derivatives with respect to all its five variables. If the initial function $u_1(x, y)$ is chosen such that it has continuous second derivatives, then we see that the function $f(x, y) = P(x, y, u_1, (\frac{\partial u_1}{\partial x}, \frac{\partial u_1}{dy})$ will have first derivatives continuous in D and on C, and therefore $u_2(x, y)$ will have second derivatives continuous in D, so that $P(x, y, u_2, p_2, q_2)$ will have first derivatives continuous in D.

We then calculate v_3 from the equation

$$\Delta^2 v_3 = P(x, y, u_2, p_2, q_2) - P(x, y, u_1, p_1, q_1). \tag{21}$$

We have just proved that the right hand member of (21) has first derivatives continuous in D. On account of § 3·1(3) this is sufficient in order that $v_3 (x, y)$ may have second derivatives.

Suppose F is the maximum modulus of the right hand member of (21), and F_1 that of its first derivative, then from (3), (5) and (8) § 3·1(3) we obtain

$$|v_3| < MF, \left|\frac{\partial v_3}{\partial x}\right| < NF, \left|\frac{\partial^2 v_3}{dx^2}\right| < \lambda F + M_1 F_1. \qquad \ldots\ldots \quad (22)$$

We go over to v_4 which satisfies the equation

$$\nabla^2 v_4 = P(x, y, u_3, p_3, q_3) - P(x, y, u_2, p_2, q_2), \quad (23)$$

where we know from (13) and (15) that the right hand member has the maximum modulus RF.

Let

$$A' = \text{Max}\left|\frac{\partial P}{\partial u}\right|, B' = \text{Max}\left|\frac{\partial P}{\partial p}\right|, C' = \text{Max}\frac{dP}{\partial q}. \qquad \ldots\ldots \quad (24)$$

Then

$$\left|\frac{\partial}{\partial x}\{P(x, y, u_3, p_3, q_3) - P(x, y, u_2, p_2, q_2)\}\right|$$

$$\leqslant \left|\left(A'\frac{\partial u_3}{\partial x} + B'\frac{\partial^2 u_3}{\partial x^2} + C'\frac{\partial^2 u_3}{\partial x \partial y}\right) - \left(A'\frac{\partial u_2}{\partial x} + B'\frac{\partial^2 u_2}{\partial x^2} + C'\frac{\partial^2 u_2}{\partial x \partial y}\right)\right|$$

$$\leqslant A'\left|\frac{\partial v_3}{\partial x}\right| + B'\left|\frac{\partial^2 v_3}{\partial x^2}\right| + C'\left|\frac{\partial^2 v_3}{\partial x \partial y}\right|$$

$$< A'NF + B'(\lambda F + M_1 F_1) + C'(\lambda F + M_1 F_1)$$

$$< (\lambda A'' + NA')F + M_1 A'' F_1, \qquad \ldots\ldots \quad (25)$$

where

$$A'' = B' + C'. \qquad \ldots\ldots \quad (26)$$

Thus from (23) we obtain

$$|v_4| < MRF, \left|\frac{\partial v_4}{\partial x}\right| < NRF, \left|\frac{\partial^2 v_4}{dx^2}\right| < \lambda RF$$
$$+ M_1(aF + bF_1),$$

where

$$a = \lambda A'' + NA', b = M_1 A''. \quad \ldots\ldots \quad (27)$$

Proceeding thus step by step, we see that the maximum moduli of $v_n, \frac{\partial v_n}{\partial x}, \frac{\partial^2 v_n}{\partial x^2}$ will be obtained from those of $v_3, \frac{\partial v_3}{\partial x}, \frac{\partial^2 v_3}{\partial x^2}$ successively by the linear substitution of

$$RF \text{ for } F, \quad aF + bF_1 \text{ for } F_1. \quad \ldots\ldots \quad (28)$$

We must make the substitution n times if we wish to obtain the maximum moduli of $v_{n+3}(x,y)$, $\frac{\partial v_{n+3}}{\partial x}, \frac{\partial^2 v_{n+3}}{\partial x^2}$. Thus the maximum modulus of $\frac{\partial^2 v_{n+3}}{dx^2}$ will be

$$\lambda R^n F + M_1\{a(R^n + R^{n-1}b + R^{n-2}b^2$$
$$+ \ldots + b^n)F + b^{n+1}F_1\}. \quad \ldots\ldots \quad (29)$$

Now R and b contain M, N or M_1 as factors, and each of these tends to zero with D. Thus taking D sufficiently small we can make both R and b less than unity, so that we find

$$\frac{\partial^2 v_n}{\partial x^2} \to 0 \text{ when } n \to \infty. \qquad \ldots\ldots \quad (30)$$

This shows that $\frac{\partial^2 u}{\partial x^2}$ exists and is continuous in D. Similar results hold for $\frac{\partial^2 u}{\partial y^2}$.

We conclude therefore that the function $u(x, y)$ given by (20) satisfies the given differential equation (1), and since it has been shown to satisfy the boundary conditions also, it is the required solution of our problem, provided that the domain D is sufficiently small.

3.3. Uniqueness of the solution.

Suppose $u_1(x, y)$ and $u_2(x, y)$ are two solutions of the differential equation

$$\nabla^2 u = P(x, y, u, p, q), \qquad \ldots\ldots \quad (1)$$

which are regular in D and take the same boundary values on C. Let

$$v(x, y) = u_2(x, y) - u_1(x, y). \qquad \ldots\ldots \quad (2)$$

Then $v(x, y)$ satisfies the differential equation

$$\begin{aligned} \nabla^2 v &= P(x, y, u_2, p_2, q_2) - P(x, y, u_1, p_1, q_1) \\ &= \frac{\partial P}{\partial u}(u_2 - u_1) + \frac{\partial P}{\partial p}(p_2 - p_1) + \frac{\partial P}{\partial q}(q_2 - q_1) \\ &= \frac{\partial P}{\partial u} v + \frac{\partial P}{\partial p}\frac{\partial v}{\partial x} + \frac{\partial P}{\partial q}\frac{\partial v}{\partial y}, \qquad \ldots\ldots \quad (3) \end{aligned}$$

where u lies between $u_1(x, y)$ and $u_2(x, y)$.

We see also that v is regular in D and vanishes identically on C.

We shall prove now that v will vanish identically in D provided D is sufficiently small, and that this zero solution is the only solution of the equation (3) taking zero values on C.

We assume that the function P and all its derivatives occurring below are continuous. We define three functions d, e, f by

$$d(x,y) = -\tfrac{1}{2}\frac{\partial P}{\partial p},\ e(x, y) = -\tfrac{1}{2}\frac{\partial P}{\partial q},\ f(x, y) = -\frac{\partial P}{\partial u}. \qquad \ldots\ldots \quad (4)$$

The equation (3) then becomes

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+2d\frac{\partial v}{\partial x}+2e\frac{\partial v}{\partial y}+fv=0. \qquad \ldots\ldots \quad (5)$$

If $v(x,y)$ is a solution of (5) in D, then evidently

$$\iint_D v\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+2d\frac{\partial v}{\partial x}+2e\frac{\partial v}{\partial y}+fv\right\}dx\,dy=0 \qquad \ldots\ldots \quad (6)$$

Integrating (6) by parts and remembering that $v=0$ on C, we get

$$\iint_D\left\{\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+v^2\ \frac{\partial d}{\partial x}+\frac{\partial e}{\partial y}-f\ \Big)\right\}dxdy=0. \qquad \ldots\ldots \quad (7)$$

Let

$$\theta(x, y) = \frac{\partial d}{dx} + \frac{\partial e}{\partial y} - f$$

$$= \frac{\partial P}{\partial u} - \frac{1}{2}\frac{\partial^2 P}{\partial x \partial p} - \frac{1}{2}\frac{\partial^2 P}{\partial y \partial q}. \qquad \ldots\ldots \quad (8)$$

If $\theta(x, y)$ is positive in D, i.e., if

$$\frac{\partial P}{\partial u} - \frac{1}{2}\frac{\partial^2 P}{\partial x \partial p} - \frac{1}{2}\frac{\partial^2 P}{\partial y \partial q} > 0 \qquad \ldots\ldots \quad (9)$$

n D, then from (7) we deduce that for all x, y in D

$$v = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0. \qquad \ldots\ldots \quad (10)$$

This means that $v(x, y)$ vanishes identically in D, so that $u_2(x, y) \equiv u_1(x, y)$ in D. The uniqueness of the solution is therefore established for any arbitrary D.

In the general case when the inequality (9) is not satisfied, we employ the following procedure to establish the uniqueness for sufficiently small D.

Let $\varphi(x, y)$ and $\psi(x, y)$ be two arbitrary real functions which are continuous in D and on C.

Then since $v = 0$ on C, we have

$$\iint_D \left\{ \frac{\partial}{\partial x}(\varphi v^2) + \frac{\partial}{\partial y}(\psi v^2) \right\} dx\, dy = 0. \qquad \ldots\ldots \quad (11)$$

Adding (7) and (11) we obtain

$$\iint_D \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\varphi v \frac{\partial v}{\partial x} + 2\,\psi v \frac{\partial v}{\partial y}\right.$$

$$\left. + v^2\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} + \frac{\partial d}{\partial x} + \frac{\partial e}{\partial y} - f\right)\right\} dxdy = 0. \quad (12)$$

The integrand is a quadratic form in $v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$, and will be definite if

$$\varphi^2 + \psi^2 < \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial x} + \theta. \quad \ldots\ldots \quad (13)$$

We shall prove that if $\theta(x, y)$ is continuous, we can determine two functions $\varphi(x, y)$ and $\psi(x, y)$ such that in the neighbourhood of an arbitrary point Q, the relation

$$\varphi^2 + \psi^2 - \theta < \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} \quad \ldots\ldots \quad (14)$$

is satisfied.

To prove this suppose that m^2 is the maximum modulus of $-\theta$. We wish to determine two functions φ and ψ which satisfy

$$\varphi^2 + \psi^2 + m^2 < \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}. \quad \ldots\ldots \quad (15)$$

The inequality (15) implies necessarily the inequality (14). We can easily find one pair φ, ψ

which satisfies (15). Thus put $\psi = 0$, and take φ a function of x only, satisfying the equation

$$\frac{\partial \varphi}{\partial x} - \varphi^2 = m_1^2, \qquad \ldots\ldots \quad (16)$$

where $m_1^2 > m^2$. The solution of (16) is

$$\varphi = m_1 \tan(m_1 x + \alpha), \qquad \ldots\ldots \quad (17)$$

where α is a constant.

Thus we can certainly determine φ in an interval between two parallels to the y-axis whose mutual distance is less than $\frac{\pi}{m_1}$, and this φ will satisfy the required relation. For this pair φ, ψ the inequality (15) and consequently the inequality (13) will be satisfied.

Hence, in a region of the plane round the point Q, for which we can determine the continuous functions φ, ψ satisfying the inequality (13), we see that the equation (12) will be possible if and only if

$$v = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0. \qquad \ldots\ldots \quad (18)$$

This shows that provided D is sufficiently small, the differential equation (3) can have no solution, other than zero, which is regular in D and which vanishes on C.

Consequently, there is one and only one solution of (1) which is regular in D and which takes given boundary values on C, provided D is sufficiently small.

The restriction that D should be small, can be removed if, instead of the general equation (1), we consider the particular non-linear equation

$$\nabla^2 u = P(x, y, u), \qquad \ldots\ldots \quad (19)$$
$$\frac{\partial P}{\partial u} \geqslant 0.$$

It can be shown* that for any arbitrarily large domain D, the equation (19) has one and only one solution which is regular in D, and which takes given values on C.

*P. Frank u. R. von Mises: Die Differential-und Integral gbichungen der Mechanik und Physik. vol. I, Chapter 18, § 2 and § 3. Also E. Picard: Journal de Math. (4 series) Vol. 6 (1890), Chap. III, p. 173.

CHAPTER IV

NON-LINEAR PARABOLIC EQUATIONS II

4·1. Resumé of the results for linear parabolic equations.

4·1 (1). Fundamental solution.

Consider the equation

$$\delta u \equiv \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \qquad \ldots\ldots \quad (1)$$

in the domain D bounded by a portion $A_1 A_2$ of the x-axis and the two curves C_1 and C_2 given by the equations

$$C_1 : \; x = h_1(t), \quad C_2 : \; x = h_2(t), \qquad \ldots\ldots \quad (2)$$

and bounded above by any parallel $B_1 B_2$ to the x-axis. Without restricting the generality, we can assume that the time $A_1 A_2$ is given by $t = 0$.

The equation which is adjoint to (1) is given by

$$\delta_1 v \equiv \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial t} = 0. \qquad \ldots\ldots \quad (3)$$

Let P and Q be any two points in D whose

coordinates are (x, t) and (ξ, η) respectively; then the fundamental solution is

$$U(\xi, \eta; x, t) = \frac{1}{\sqrt{t-\eta}} e^{-\frac{(x-\xi)^2}{4(t-\eta)}}. \qquad \ldots\ldots \quad (4)$$

It can be easily verified that $U(\xi, \eta; x, t)$ is a solution of $\delta u = 0$ considered as a function of x, t, and of $\delta_1 v = 0$ considered as a function of ξ, η.

Let x_1 and x_2 be the abscissae of A_1 and A_2 respectively, and consider the integral

$$K(x, t) = \int_{x_1}^{x_2} \frac{1}{\sqrt{t}} e^{-\frac{(x-\xi)^2}{4t}} \psi(\xi)\, \alpha\xi, \qquad \ldots\ldots \quad (5)$$

where $\psi(x)$ is a function continuous in (x_1, x_2).

This integral is called 'Poisson's formula,' and it represents a solution of the equation $\delta u = 0$, which is regular in D except on the characteristic $A_1 A_2$. It vanishes at all points below this characteristic, and also at all points of this characteristic outside the segment $A_1 A_2$. In the portion of the plane situated above $A_1 A_2$ it is a holomorphic function of x, t.

We have further

$$\left.\begin{aligned} \lim_{(x,t)\to(x_0,0)} K(x, t) &= 2\sqrt{\pi}\, \psi(x_0), \text{ if } x_0 \text{ is in } (x_1, x_2) \\ &= 0, \text{ if } x_0 \text{ is outside } (x_1, x_2). \end{aligned}\right\} \qquad \ldots\ldots \quad (6)$$

4·1 (2). Boundary problem for $\delta u = 0$.

With the help of a system of two linear integral equations it can be shown that one and only one solution of $\delta u = 0$ exists which vanishes on $A_1 A_2$, and satisfies on C_1, C_2 relations of the form

$$\lambda_i(t) \frac{\partial u}{\partial x} + \mu_i(t) u = \nu_i(t), \; (i = 1, 2). \quad \ldots\ldots \quad (1)$$

It is necessary for the truth of this existence theorem that the curves C_1 and C_2 are rectifiable, and that their equations satisfy the inequalities

$$| h_i(t) - h_i(\eta) | < L \, | t - \eta |^{\frac{1+\alpha}{2}}, \; (0 < \alpha \leqslant 1) \quad (2)$$

where L and α are two constants, and $i = 1, 2$.

The following two properties of this solution of $\delta u = 0$ are analogous to those for the potential functions.

1. If a solution is regular in D and continuous on (C), it cannot surpass its extreme values which it attains on (C).

2. If a sequence of solutions converges uniformly on (C), it will converge uniformly in D and will represent a solution.

4·1 (3). Green's function.

Through the point P with coordinates (x, t)

draw a line parallel to the x-axis cutting the curves C_1 and C_2 at M_1 and M_2 respectively. Let the domain bounded by $M_1 A_1 A_2 M_2 M_1$ be called D_t.

Suppose $H(\xi, \eta; x, t)$ is a solution of the equation

$$\delta_1 H \equiv \frac{\partial^2 H}{\partial \xi^2} + \frac{\partial H}{\partial \eta} = 0, \qquad \ldots\ldots \quad (1)$$

which is regular in D_t, vanishes at each point of the segment $M_1 M_2$ and takes the same values as the function $\frac{1}{\sqrt{t-\eta}} e^{-\frac{(x-\xi)^2}{4(t-\eta)}}$ on the curves $M_1 A_1$ and $M_2 A_2$.

Then

$$G(\xi, \eta; x, t) \equiv U(\xi, \eta; x, t) - H(\xi, \eta; x, t) \quad (2)$$

is called the "Green's function" of $\delta u = 0$ for (C). It is the solution of $\delta u = 0$ in x, t, and of $\delta_1 v = 0$ in ξ, η. It vanishes when the point (ξ, η) [or the point (x, y)] is on C_1 or C_2, and behaves in the neighbourhood of the point (x, t) [or of (ξ, η)] as the fundamental solution U. It is always positive in D.

It is easily seen that the Green's function is symmetrical in (x, t) and (ξ, η):

$$G(x, t; \xi, \eta) = G(\xi, \eta; x, t). \qquad \ldots\ldots \quad (3)$$

Further it can be shown that

$$G = g\, U(\xi, \eta; x, t), \ (0 \leqslant g \leqslant 1) \quad \ldots\ldots \quad (4)$$

where g is a bounded function lying between zero and 1.

Again

$$\frac{\partial G}{\partial x} = \frac{\partial U}{\partial x} - \frac{\partial H}{\partial x}, \qquad \ldots\ldots \quad (5)$$

from which we see that $\dfrac{\partial G}{\partial x} = 0$ when (ξ, η) is on C_1 or C_2 and (x, t) is in D. Similarly all derivatives of G with respect to x and t vanish when (ξ, η) is on C_1 or C_2 and (x, t) is in D. Conversely, all derivatives of G with respect to ξ and η vanish when (x, t) is on C_1 or C_2 and (ξ, η) is in D.

4·1 (4). The fundamental formula.

Let $\varphi(x, t)$ and $\psi(x, t)$ be two functions having second derivatives. Then we have

$$\psi\delta\varphi - \varphi\delta_1\psi = \psi\left(\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial\varphi}{\partial t}\right) - \varphi\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial t}\right)$$

$$= \frac{\partial}{\partial x}\left(\psi\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial t}(\varphi\psi).$$

Hence from Gauss' theorem we get

$$\iint_D (\psi\delta\varphi - \varphi\delta_1\psi)\, dxdt = \int_{(C)} \varphi\psi\, dx + \left(\psi\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial\psi}{\partial x}\right) dt. \qquad \ldots\ldots \quad (1)$$

This is called the extended Green's formula, or Riemann's formula.

From (1) we get, if $u(x, t)$ is a solution of the equation $\delta u = f(x, t)$,

$$\left.\begin{array}{ll}(\alpha) & 2\sqrt{\pi}\, u(x, t) \\ (\beta) & \sqrt{\pi}\, u(x, t) \\ (\gamma) & 0\end{array}\right\} = \int_{(M_1A_1M_2A_2)} \frac{1}{\sqrt{t-\eta}} e^{-\frac{(x-\xi)^2}{4(t-\eta)}} \Big\{ u(\xi, \eta)\, d\xi$$

$$+ \left(\frac{\partial u}{\partial \xi} - u(\xi, \eta) \frac{x-\xi}{2(t-\eta)}\right) d\eta \Big\}$$

$$- \iint_{D_t} \frac{1}{\sqrt{t-\eta}} e^{-\frac{(x-\xi)^2}{4(t-\eta)}} f(\xi, \eta)\, d\xi d\eta. \qquad (2)$$

This is the "fundamental formula" in which (α) holds when (x, t) is inside D, (β) holds when (x, t) is on (C), and (γ) holds when (x, t) is outside D*,

4·1 (5). The first boundary problem for $\delta u = 0$.

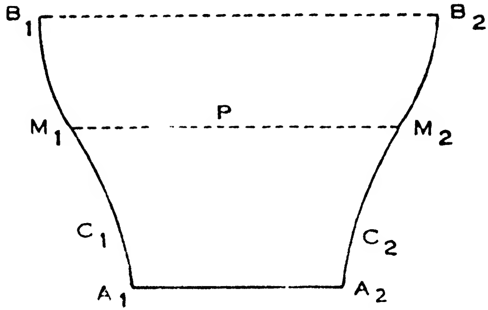
We wish to determine a solution of $\delta u = 0$ which is regular in D, and which takes the following boundary values on (C):

$$u = \varphi_1(t) \text{ on } C_1, \;\; u = \varphi_2(t) \text{ on } C_2, \qquad (1)$$

* For details see Goursat, Course d'Analyse, Vol. 3. § 544 pp. 308 *et seq.*

and

$$u = \varphi(x) \text{ on } A_1A_2. \qquad \ldots\ldots \quad (2)$$



Let

$$u(x, t) = \frac{1}{2\sqrt{\pi}}\int_{a_1}^{a_2} \frac{1}{\sqrt{t}} e^{-\frac{(x-\xi)^2}{4t}} \psi(\xi)\, d\xi, \quad \ldots \quad (3)$$

where a_1 and a_2 are two numbers such that $a_1 < x_1 < x_2 < a_2$, and where $\psi(x)$ is a function continuous in (a_1, a_2) coinciding with $\varphi(x)$ in (x_1, x_2). From § 4·1 (1) we see that $\delta \bar{u} = 0$ in D, and that $\bar{u} = \varphi(x)$ on A_1A_2.

Suppose $u(x, t)$ is a solution of the problem (1), (2), and write

$$u(x, t) = \bar{u}(x, t) + u_1(x, t). \qquad \ldots\ldots \quad (4)$$

Then $u_1(x, t)$ satisfies the equation

$$\delta u_1 = 0, \qquad \ldots\ldots \quad (5)$$

and the boundary conditions

$$u_1 = \varphi_1(t) \text{ on } C_1, \quad u_1 = \varphi_2(t) \text{ on } C_2; \quad \ldots\ldots \quad (6)$$

$$u_1 = 0 \text{ on } A_1 A_2, \quad \ldots\ldots \quad (7)$$

provided of course that $\varphi_1(0) = \varphi_2(0) = 0$.

To find u_1 we have from the fundamental formula (2α) § 4·1 (4), on writing $f = 0$,

$$u_1(x, t) = \frac{1}{2\sqrt{\pi}} \int_{(M_1 A_1 + A_2 M_2)} \frac{1}{\sqrt{t-\eta}} e^{-\frac{(x-\xi)^2}{4(t-\eta)}}$$

$$\left\{ u_1\, d\xi + \left(\frac{\partial u_1}{\partial \xi} - u \frac{x-\xi}{2(t-\eta)} \right) \right\} d\eta. \quad \ldots\ldots \quad (8)$$

Similarly, if in (1) § 4·1 (4) we substitute $u_1(\xi, \eta)$ for φ and $H(\xi, \eta;\, x, t)$ for ψ, and remark that $u_1 = 0$ on $A_1 A_2$ and $H = 0$ on $M_1 M_2$, we get

$$0 = \frac{1}{2\sqrt{\pi}} \int_{(M_1 A_1 + A_2 M_2)} u_1 H \, d\xi$$

$$+ \left(H \frac{\partial u_1}{\partial \xi} - u_1(\xi, \eta) \frac{\partial H}{\partial \xi} \right) d\eta. \quad \ldots\ldots \quad (9)$$

Subtracting (9) from (8), and remembering that H has the same values as U on C_1 and C_2, we get

$$u_1(x, t) = -\frac{1}{2\sqrt{\pi}} \int_{(M_1 A_1 + A_2 M_2)} u_1(\xi, \eta)$$

$$\frac{\partial G^{(\xi,\, \eta:\, x,\, t)}}{\partial \xi} d\eta, \quad \ldots\ldots \quad (10)$$

The values of u_1 are given on the curves C_1, C_2, so that the integral (10) gives us $u_1(x, t)$ in D. Hence on account of (4) the required solution $u(x, t)$ is obtained.

Any function $f(x)$ is said to admit an increment of order γ (non-zero) if for any two values x_1 and $x_2 (> x_1)$ of x, we have

$$|f(x_2) - f(x_1)| < L\, |x_2 - x_1|^{\gamma} \quad \ldots\ldots \quad (11)$$

where L and γ are constants, and γ is positive. $f(x)$ is called by some writers to be continuous (L).

Thus if $\varphi_1'(t)$, $\varphi_2'(t)$, $\varphi'(x)$, $\varphi''(x)$, $h_1'(t)$, $h_2'(t)$ exist and admit increments of order non-zero at all points of D and (C), then $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial u}{\partial t}$ exist, and we can determine a number $\gamma < 1$ such that for a given increment Δt of t we obtain

$$|\Delta u| < L\, |\Delta t|, \quad \left|\Delta \frac{\partial u}{\partial x}\right| < L\, |\Delta t|^{\gamma+\beta},$$

$$\left|\Delta \frac{\partial u}{\partial t}\right| < L\, |\Delta t|^{\gamma}, \quad \ldots\ldots \quad (12)$$

where L is a constant, and where β is the smaller of the two numbers $\frac{1}{2}$ and $1 - \gamma$.

All the results of this section hold also for the equation $\delta u = f(x)$

4.1 (6). The equation $\delta u = f(x, t)$.

We consider now the equation

$$\delta u \equiv \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f(x, t), \qquad \ldots\ldots \quad (1)$$

and determine the solution $u(x, t)$ which is regular in D, and takes the same boundary values as (1) (2) of § 4.1 (5). Then by a method similar to that employed in the previous section we get for the solution of this boundary problem:

$$u(x, t) = -\frac{1}{2\sqrt{\pi}}\int_{(M_1A_1+A_2M_2)} u(\xi, \eta)\frac{\partial G^{(\xi, \eta; x, t)}}{\partial \xi} d\eta$$

$$+\frac{1}{2\sqrt{\pi}}\int_{A_1A_2} G(\xi, 0; x, t)\, u(\xi, 0)\, d\xi$$

$$-\frac{1}{2\sqrt{\pi}}\iint_{D_t} G(\xi, \eta; x, t)\, f(\xi, \eta)\, d\xi\, d\eta. \quad (2)$$

If the function $f(x, t)$ has the maximum modulus F in D, and if $\varphi(x)$, $\varphi'(x)$ have maximum moduli M, M' respectively in (x_1, x_2), then we find in D

$$|u| < M + Ft, \left|\frac{\partial u}{\partial x}\right| < M' + \frac{4}{\sqrt{\pi}}\sqrt{t} \qquad (3)$$

But if $|f| < Ft^\gamma$, (F constant) then we obtain

$$|u| < M + F\frac{t^{\gamma+1}}{\gamma+1}, \quad \left|\frac{\partial u}{\partial x}\right| < M' + L't^\beta,$$

$$(0 < \beta < 1), \qquad \ldots\ldots \quad (4)$$

where L' is a number depending on the increments of $\varphi_i(t)$, $\varphi'(x)$ and $h_i(t)$, and β is smaller of the two numbers $\frac{1}{2}$ and $1-\gamma$.

Now suppose that $u_0(x, t)$ is a solution of $\delta u = f$ vanishing on (C), so that on account of (2) we have

$$u_0(x, t) = -\frac{1}{2\sqrt{\pi}} \iint_{D_t} G(\xi, \eta\,;\, x, t) f(\xi, \eta) d\xi\, d\eta. \quad \ldots\ldots \quad (5)$$

Then it is found that if $|f| < F\, t^{\gamma}$ in D.

$$|u_0| < F \frac{t^{\gamma+1}}{\gamma+1}, \quad \left|\frac{\partial u_0}{\partial x}\right| < LFB\left(\tfrac{1}{2}, \gamma+1\right) t^{\gamma+1/2}, \quad \ldots\ldots \quad (6)$$

where B denotes the Beta function.

If $(f(x, t)$ satisfies the condition

$$|\Delta f| = |f(x, t+\Delta t) - f(x, t)| < K(\Delta t)^{\gamma},$$
$$f(x, 0) = 0, \quad \ldots\ldots \quad (7)$$

K being a constant, then we obtain

$$\left.\begin{aligned} |\Delta u_0| < LK\Delta t, \quad \left|\Delta \frac{\partial u_0}{\partial x}\right| &< LK(\Delta t)^{\gamma+\beta}, \\ \left|\Delta \frac{\partial u_0}{\partial y}\right| &< LK(\Delta t)^{\gamma}, \\ |u_0| < LK\, t^{\gamma+1}, \left|\frac{\partial u_0}{\partial x}\right| LK\, t^{\gamma+1/2}, \left|\frac{\partial u_0}{\partial y}\right| &< LKt^{\gamma}, \end{aligned}\right\} \quad (8)$$

where L is a constant.

4·2. The non-linear equation $\delta u = P(x, t, u, p)$

4·2 (1). Existence theorem.

Consider the equation

$$\delta u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = P(x, t, u, p), \left[p \equiv \frac{\partial u}{\partial x}\right] \quad (1)$$

where the function P remains finite and well determined when $|u| < N$ and $|p| < N$, provided (x, t) remains in the domain D. P is a continuous function of x, t, satisfying the conditions

$$\left.\begin{aligned} |P(\xi, \eta) - P(\xi, t)| &< K|t - \eta|^\gamma, \\ |P(\xi, \eta) - P(x, \eta)| &< K|x - \xi|^\gamma, \end{aligned}\right\} (0 < \gamma \leqslant 1) \quad (2)$$

P satisfies also the Lipschitz condition in u, p, i.e.,

$$\begin{aligned} |P(x, t, u_2, p_2) - P(x, t, u_1, p_1)| \\ < A|u_2 - u_1| + C|p_2 - p_1| \quad (3) \end{aligned}$$

We wish to determine a solution $u(x, t)$ of (1) which is regular in D and which satisfies the following boundary conditions:

$$u = \varphi_1(t) \text{ on } C_1 \text{ and } u = \varphi_2(t) \text{ on } C_2, \quad (4)$$

and

$$u(x, 0) = \varphi(x), \text{ i.e., } u = \varphi(x) \text{ on } A_1 A_2. \quad (5)$$

The curves C_1 and C_2 are such that their equations satisfy the inequalities (2) § 4·1 (2). $\varphi'(x)$ exists and admits of an increment of order non-zero; $\varphi_1(t)$ and $\varphi_2(t)$ admit of increments of order $> \frac{1}{2}$.

Let $u_0(x, t)$ be any arbitrary function such that $|u_0| < N$ and $|p_n| < N$, where we write: $p_n = \frac{\partial u_n}{\partial x}$ $(n = 0, 1, 2, 3, \ldots)$. Then determine successively the solution of the following equations:

$$\delta u_1 = P(x, t, u_0, p_0), \qquad \ldots\ldots \quad (6)$$

$$\delta u_2 = P(x, t, u_1, p_1), \qquad \ldots\ldots \quad (7)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\delta u_n = P(x, t, u_{n-1}, p_{n-1}), \qquad \ldots\ldots \quad (8)$$

$u_1, u_2, \ldots, u_n, \ldots$ taking the given values (4), (5) on (C).

Since we know $u_0(x, t)$, the right hand member of (6) is a known function of (x, t). From 4·1(6) we can therefore find a solution of (6) satisfying the boundary conditions. Moreover, since $|u_0|$, $|p_0| < N$, the function $P(x, t, u_0, p_0)$ has a maximum modulus F. On account of (3) and (4) § 4·1 (6) we get therefore

$$|u_1| < M + Ft, \quad |p_1| < M' + L' t^{\beta}, \qquad \ldots\ldots \quad (9)$$

M, M', L' β having the same significance as in § 4·1 (6).

Now, u_1 and p_1, must not lie outside the interval $(-N, N)$, so that we must have

$$\left.\begin{array}{ll} M+Ft<N, & M'+L't^{\beta}<N \\ \text{or} \quad t<\dfrac{N-M}{F}, & t<\left(\dfrac{N-M'}{L'}\right)^{\frac{1}{\beta}}. \end{array}\right\} \qquad (10)$$

t must be therefore less than the smaller of these two numbers if $|u_1|$ is to be less than N. for this sufficiently small domain we then solve the equation (7) and get the same inequalities (9) for u_2, p_2, which on account of (10) gives us again $|u_2|<N$, $|p_2|<N$. Similarly for u_3, p_3 and other approximations. Thus we are assured that for the sufficiently small domain (10) u_n, p_n remain inside the interval $(-N, N)$.

Now we write

$$v_n(x,t)=u_{n+1}(x,t)-u_n(x,t),\ (n=1,2,3,\ldots), \qquad \ldots\ldots \quad (11)$$

and get from (8) and (11):

$$\delta v_1 = P(x,t,u_1,p_1)-P(x,t,u_0,p_0) \quad \ldots\ldots \quad (12)$$

$$\delta v_2 = P(x,t,u_2,p_2)-P(x,t,u_1,p_1) \quad \ldots\ldots \quad (13)$$

. . . .

$$\delta v_n = P(x,t,u_n,p_n)-P(x,t,u_{n-1},p_{n-1}), \qquad (14)$$

. . . .

Evidently all $v_n(x,t)$ vanish on C.

Since P satisfies the Lipschitz condition (3), we get

$$|\delta v_1| < A|u_1 - u_0| + C|p_1 - p_0|, \quad \ldots\ldots \quad (15)$$

$$|\delta v_2| < A|v_1| + C\left|\frac{\partial v_1}{\partial x}\right|, \quad \ldots\ldots \quad (16)$$

....

$$|\delta v_n| < A|v_{n-1}| + C\left|\frac{\partial v_{n-1}}{\partial x}\right|. \quad \ldots\ldots \quad (17)$$

Since $|u_0|$, $|p|_0$, $|u_1|$, $|p_1| < N$, we find from (15) that the right hand member of (12) has the maximum modulus M_0 such that

$$M_0 < 2N(A + C). \quad \ldots\ldots \quad (18)$$

Then we get from (6) § 4·1 (6) on writing $\gamma = 0$ and $F = M_0$:

$$|v_1|, \left|\frac{\partial v_1}{\partial x}\right| < \mathrm{L}\, M_0\, B(\tfrac{1}{2}, 1)\, t^{1/2}$$

$$< 2N\frac{\mathrm{L}(A + C)\sqrt{\pi t}}{\Gamma(\frac{3}{2})} = M_1 \text{ (say)}. \quad \ldots\ldots \quad (19)$$

From (16) we see that the right hand member of (13) has a maximum modulus $M_1(A + C)$, and therefore for the solution v_2 we find:

$$|v_2|, \left|\frac{\partial v_2}{\partial x}\right| < \mathrm{L}.\ M_1(A + C).\ B(\tfrac{1}{2}, 1 + \tfrac{1}{2})\, t^{1/2}$$

$$< 2N.\frac{\{\mathrm{L}(A + C)\sqrt{\pi t}\}^2}{\Gamma(2)}. \quad \ldots\ldots \quad (20)$$

Proceeding thus step by step, we get in general

$$|v_n|,\ \left|\frac{\partial v_n}{\partial x}\right| < \frac{2N}{\Gamma\left(\frac{n+2}{2}\right)}\{L(A+C)\sqrt{\pi t}\}^n. \qquad (21)$$

We assume the domain to be so small that

$$L(A+C)\sqrt{\pi t} < 1, \text{ i.e. } t < \frac{1}{L^2(A+C)^2}, \qquad (22)$$

Then v_n, $\frac{\partial v_n}{\partial x}$ will tend uniformly to zero, and consequently u_n will converge uniformly to a limit $u(x, t)$ represented by the infinite series:

$$u(x, t) = u_1 + v_1 + v_2 + v_3 + \dots + v_n + \dots \qquad (23)$$

This function $u(x, t)$ is then the required solution of our problem.

4·2 (2) Uniqueness of the solution.

Suppose the equation

$$\delta u = P(x, t, u, p) \qquad \ldots\ldots \quad (1)$$

has two solutions $u_1(x, t)$ and $u_2(x, t)$ which are regular in D and take the same boundary values (4), (5) of § 4.2(1) on (C). We write

$$v(x, t) = u_2(x, t) - u_1(x, t). \qquad \ldots\ldots \quad (2)$$

and see that v is regular in D, vanishes identically

on (C), and satisfies the differential equation

$$\delta v = P(x, t, u_2, p_2) - P(x, t, u_1, p_1)$$
$$= g(x, t), \text{ (say).} \qquad \ldots\ldots \quad (3)$$

Let M be the maximum modulus of $g(x, t)$ in D, then we get from (6) § 4.1(6):

$$|v| \text{ and } \left|\frac{\partial v}{\partial x}\right| < \mathrm{L}\, MB(\tfrac{1}{2}, 1)\, t^{1/2}. \qquad \ldots\ldots \quad (4)$$

But since P is Lipschitzian, we get

$$|g(x, t)| < A\,|v| + C\left|\frac{\partial v}{\partial x}\right| \qquad \ldots\ldots \quad (5)$$
$$< (A + C).\, \mathrm{L}\, MB(\tfrac{1}{2}, 1)\, t^{1/2}$$
$$= M_1 \text{ (say).} \qquad \ldots\ldots \quad (6)$$

Since the right hand member of (3) has now the maximum modulus M_1, we get

$$|v|,\ \left|\frac{\partial v}{\partial x}\right| < \mathrm{L}\, M_1\, B(\tfrac{1}{2}, 1 + \tfrac{1}{2})\, t^{1/2}$$

$$< \frac{M}{(A + C)\,\Gamma(2)}\left\{\mathrm{L}(A + C)\sqrt{\pi t}\right\}^2 = M_2 \text{ (say)} \quad (7)$$

Proceeding thus step by step, we get in general

$$|v| \text{ and } \left|\frac{\partial v}{\partial x}\right| < \frac{M}{(A + C)\,\Gamma\left(1 + \frac{n}{2}\right)}.$$
$$\{\mathrm{L}(A + C)\sqrt{\pi t}\}^n. \qquad \ldots\ldots \quad (8)$$

But $L(A+C)\sqrt{\pi t} < 1$ on account of (22) § 4.2 (1). Therefore we see that for all x, t in D the function v is identically zero, and consequently the two solutions u_1 and u_2 are identical.

4·3. The non-linear equation $\delta u = P(x, t, u, p, q)$.

We consider the equation

$$\delta u = P(x, t, u, p, q), \left[p = \frac{\partial u}{\partial x}, \; q = \frac{\partial u}{\partial t}\right], \qquad (1)$$

for the boundary values

$$u = \varphi_1(t) \text{ on } C_1, \; u = \varphi_2(t) \text{ on } C_2 \qquad \dots\dots \quad (2)$$

and

$$u(x, 0) = \varphi(x) \text{ for } x \text{ in } A_1 A_2 \qquad \dots\dots \quad (3)$$

We assume that the function $P(x, t, u, p, q)$ is continuous for x, t in D and when $|u|, |p|, |q| < N$. Under these conditions the four derivatives $\frac{\partial P}{\partial t}, \frac{\partial P}{\partial u}, \frac{\partial P}{\partial p}, \frac{\partial P}{\partial q}$ exist. We assume further that these four derivatives satisfy the Lipschitz condition, and that $\frac{\partial P}{\partial q}$ is essentially positive and bounded, so that

$$\frac{\partial P}{\partial q} < M_0. \qquad \dots\dots \quad (4)$$

The curvos C_1 and C_2 are such that $h_1'(t)$ and $h_2'(t)$ exist and admit increments of order non-zero.

Similarly, $\varphi_1'(t)$, $\varphi_2'(t)$, $\varphi'(x)$, $\varphi''(x)$ exist and admit increments of order non-zero. Moreover, $|\varphi_1(t)|$, $|\varphi_2(t)|$, $|\varphi(x)|$, $|\varphi'(x)|$ are always less than N. At the points $A_1(x_1, 0)$ and $A_2(x_2, 0)$, we must have

$$\begin{aligned}
&\varphi''(x_1) - \varphi_1'(0) - \varphi'(x_1)\, h_1'(0) \\
&\quad = P\{x_1, 0, \varphi(x_1), \varphi'(x_1), \varphi_1'(0) - \varphi'(x_1)\, h_1'(0)\}, \\
&\varphi''(x_2) - \varphi_2'(0) - \varphi'(x_2)\, h_2'(0) \\
&\quad = P\{x_2, 0, \varphi(x_2), \varphi'(x_2), \varphi_2'(0) - \varphi'(x_2)\, h_2'(0)\}.
\end{aligned} \qquad \ldots\ldots \quad (5)$$

Finally, the equation

$$\varphi''(x) - \theta = P\{x, 0, \varphi(x), \varphi'(x), \theta\} \qquad (6)$$

should define a function $\theta(x)$ lying between $-N$ and $+N$, and admitting an increment of order non-zero.

We solve the equation (1) again by the method of successive approximations, and start with a function $u_1(x, t)$ satisfying the equation

$$\delta u_1 = P\{x, 0, \varphi(x), \varphi'(x), \theta(x)\}, \quad \ldots\ldots \quad (7)$$

and taking the boundary values (2), (3), on (C).

For further approximations we set, if

$$p_n = \frac{\partial u_n}{\partial x}, \; q_n = \frac{\partial u_n}{\partial t},$$

$$\delta v_1 = P(x, t, u_1, p_1, q_1) - P(x, t, \varphi, \varphi', \theta),$$
$$u_2 = u_1 + v_1, \qquad (8)$$

$$\delta v_2 = P(x, t, u_2, p_2, q_2) - P(x, t, u_1, p_1, q_1),$$
$$u_3 = u_2 + v_2, \qquad (9)$$

. . . .

$$\delta v_n = P(x, t, u_n, p_n, q_n) - P(x, t, u_{n-1},$$
$$p_{n-1}, q_{n-1}), u_{n+1} = u_n + v_n, \qquad (10)$$

all the v_n vanishing identically on (C).

We form the function

$$u_n = u_1 + v_1 + v_2 + \dots + v_{n-1}, \qquad (11)$$

and see that u_n will be the solution of (1) taking the given values on (C).

Now from (10) we obtain

$$\delta v_n = P\left(x, t, u_{n-1} + v_{n-1}, p_{n-1} + \frac{\partial v_{n-1}}{\partial x} q_{n-1} + \frac{\partial u_{n-1}}{\partial t}\right)$$
$$- P(x, t, u_{n-1}, p_{n-1}, q_{n-1}), \qquad (12)$$

so that $\frac{\partial v_{n-1}}{\partial t}$ enters on the right side. We must therefore establish the convergence of $\sum_n \frac{\partial v_n}{\partial t}$,

for which we must find the maximum modulus of $\frac{\partial v_n}{\partial t}$. We can do this by utilising the increment of the right hand member of (12). For this it is necessary to find the increment of $\frac{\partial v_{n-1}}{\partial t}$ itself.

We write for brevity

$$\delta v_n = \psi_n, \quad \frac{\partial v_{n-1}}{\partial x} = w_{n-1}, \quad \frac{\partial v_{n-1}}{\partial t} = y_{n-1} \qquad (13)$$

We shall calculate the increment of ψ_n when t undergoes an increment $\triangle t$, and consequently when u_{n-1}, v_{n-1}, ... undergo the corresponding increments $\triangle u_{n-1}$, $\triangle v_{n-1}$, ... We shall have therefore to apply the formula for finite differences of a function ψ_n of seven variables t, u, p, q, v, w, y, where we have suppressed the index $(n-1)$. We denote by $t', u', p', q', v', w', y'$ quantities which are contained between the extreme values of t, u, p, q, v, w, y, and write

$$P^{(1)} = P(x, t', u', p', q'),$$
$$P^{(2)} = \psi(x, t', u' + v', p' + w', q' + y').$$

We shall have then

$$\triangle \psi_n = \triangle t \left(\frac{\partial P^{(2)}}{\partial t'} - \frac{\partial P^{(1)}}{\partial t'}\right) + \triangle u \left(\frac{\partial P^{(2)}}{\partial u'} - \frac{\partial P^{(1)}}{\partial u'}\right)$$

$$+\triangle p\left(\frac{\partial P^{(2)}}{\partial p'}-\frac{\partial P^{(1)}}{\partial p'}\right)+\triangle q\left(\frac{\partial P^{(2)}}{\partial q'}-\frac{\partial P^{(1)}}{\partial q'}\right)$$
$$+\triangle v\,\frac{\partial P^{(2)}}{\partial v'}+\triangle w\,\frac{\partial P^{(2)}}{\partial w'}+\triangle y\,\frac{\partial P^{(2)}}{\partial y'}. \quad (14)$$

We assume that the arguments of the function ψ always remain in the field of variation which we have defined, so that under these conditions $\frac{\partial P}{\partial q}$ has the maximum modulus M_0, and $\frac{\partial P}{\partial u}$, $\frac{\partial P}{\partial p}$ have the maximum moduli M_1. Since $\frac{\partial P}{\partial t}$, $\frac{\partial P}{\partial u}$, $\frac{\partial P}{\partial p}$, $\frac{\partial P}{\partial q}$ are assumed to be Lipschitzian in u, p, q, therefore for an increment $\triangle' u$, $\triangle' p$, $\triangle' q$ of u, p, q, the increments of the four derivatives of P are given by

$$\left|\triangle\frac{\partial P}{\partial t}\right|, \left|\triangle\frac{\partial P}{\partial u}\right|, \left|\triangle\frac{\partial P}{\partial p}\right|, \left|\triangle\frac{\partial P}{\partial q}\right|$$
$$< M_2(|\triangle' u|+|\triangle' p|+|\triangle' q|). \quad \ldots \quad (15)$$

Let $[v]$, $[w]$, $[y]$ denote the maximum moduli of v, w, y in the interval $\triangle t$, then from (14) we get, on reintroducing the index $n-1$,

$$|\triangle\psi_n| < M_1(|\triangle v_{n-1}|+|\triangle w_{n-1}|+|\triangle y_{n-1}|)$$
$$+M_2\{[v_{n-1}]+[w_{n-1}]+[y_{n-1}]\}\{|\triangle t|$$
$$+|\triangle u_{n-1}|+|\triangle p_{n-1}|+|\triangle q_{n-1}|\}. \quad \ldots \quad (16)$$

We study now the behaviour of the successive approximations, and start with the function $u_1(x, t)$ given by the equation (7), which is of the form $\delta u = f(x)$. Thus from (12) § 4.1 (5) we get for $\triangle t > 0$:

$$|\triangle u_1| < L \triangle t, \; \left|\triangle \frac{\partial u_1}{\partial x}\right| < L(\triangle t)^{\gamma+\beta},$$

$$\left|\triangle \frac{\partial u_1}{\partial t}\right| < L(\triangle t)^{\gamma}. \qquad \ldots\ldots \quad (17)$$

Suppose the domain D is sufficiently small given by the values of t in $0 \leqslant t \leqslant l$, then from (17) we get, since $\frac{\partial}{\partial t} u_1(x, 0) = \theta(x)$,

$$|u_1 - \varphi(x)| < Ll, \; |p_1 - \varphi'(x)| < Ll^{\gamma+\beta},$$

$$|q_1 - \theta(x)| < Ll^{\gamma}. \qquad \ldots\ldots \quad (18)$$

Now we take the equation (8), viz. $\delta v_1 = \psi_1$, and apply the formula (16) for $n = 1$. We also write

$$u_0 = \varphi(x), \; p_0 = \varphi'(x), q_0 = \theta(x),$$

$$v_0 = u_1 - u_0, w_0 = p_1 - p_0, y_0 = q_1 - q_0,$$

so that

$$\triangle v_0 = \triangle u_1, \triangle w_0 = \triangle p_1, \triangle y_0 = \triangle q_1, \quad (19)$$

since u_0, p_0, q_0 are independent of t. From (16) for $n = 1$ and from (17) and (19) we get therefore

$$|\triangle \psi_1| < K_1(\triangle t)^{\gamma}. \qquad \ldots\ldots \quad (20)$$

Consequently from (7) and (8) § 4.1(6) we obtain

$$\left.\begin{aligned} &|\triangle v_1| < \mu K_1 \triangle t,\ |\triangle w_1| < \mu K_1 (\triangle t)^{\gamma+\beta}, \\ &\qquad\qquad |\triangle y_1| < \mu K_1 (\triangle t)^{\gamma}. \\ &|v_1| < \mu K_1 l^{\gamma+1},\ |w_1| < \mu K_1 l^{\gamma+1/2}, \\ &\qquad\qquad |y_1| < \mu K_1 l^{\gamma}, \end{aligned}\right\} \quad (21)$$

where μ depends on the given values and the Contour. We can always choose γ so that it is the same in the formulae (17) and (21).

Suppose that in general ψ_{n-1} is zero on $A_1 A_2$, and that we have

$$|\triangle \psi_{n-1}| < K_{n-1} (\triangle t)^{\gamma}, \qquad \ldots\ldots \quad (22)$$

and that analogous to (13) we have

$$\left.\begin{aligned} &|\triangle v_{n-1}| < \mu K_{n-1} \triangle t,\ |\triangle w_{n-1}| < \mu K_{n-1} \\ &\quad \times (\triangle t)^{\gamma+\beta},\ |\triangle y_{n-1}| < \mu K_{n-1} (\triangle t)^{\gamma}, \\ &|v_{n-1}| < \mu K_{n-1} l^{\gamma+1},\ |w_{n-1}| < \mu K_{n-1} l^{\gamma+1/2}, \\ &\qquad\qquad |y_{n-1}| < \mu K_{n-1} l^{\gamma}. \end{aligned}\right\} \quad (23)$$

We can then write

$$\begin{aligned} u_{n-1} &= u_1 + v_1 + v_2 + \ldots + v_{n-2}, \\ S_{n-2} &= K_1 + K_2 + \ldots + K_{n-2}, \end{aligned} \qquad (24)$$

so that from (23) we get

$$|u_{n-1} - u_1| < \mu S_{n-2} l^{\gamma+1},\ |p_{n-1} - p_1| < \mu S_{n-2} l^{\gamma+1/2},\ |q_{n-1} - q_1| < \mu S_{n-2} l^{\gamma}. \quad (25)$$

From (17) and (25) we find then

$$| \triangle u_{n-1} | < (L + \mu S_{n-2}) \triangle t,$$
$$| \triangle p_{n-1} | < (L + \mu S_{n-2}) (\triangle t)^{\gamma+\beta},$$
$$| \triangle q_{n-1} | < (L + \mu S_{n-2}) (\triangle t)^{\gamma}. \quad \ldots\ldots \quad (26)$$

Consequently on account of (16) $\triangle \psi_n$ will be of the order γ, and we can write

$$| \triangle \psi_n | < K_n (\triangle t)^{\gamma}, \quad \ldots\ldots \quad (27)$$

where K_n is obtained by the relation

$$\frac{K_n}{K_{n-1}} = 2\mu M_1 (\triangle t)^{\beta} + M_0 \mu + \mu M_2 l^{\gamma}$$
$$\times (1 + \sqrt{l} + l) \{ (\triangle t)^{\beta} + (L + \mu S_{n-2}) (2 \triangle t^{\beta} + 1) \}$$
$$\ldots\ldots \quad (28)$$

Thus the inequalities (21) would hold for the index n with the same γ, and if the expression (28) is less than 1, the series $\sum\limits_n K_n$ will be convergent, and consequently the series $\sum\limits_n v_n$, $\sum\limits_n w_n$, $\sum\limits_n y_n$ will also be convergent.

Hence it is necessary that we should be able to determine an upper limit to the expression (28) which may be equal to $\varrho < 1$.

In this case

$$S_{n-2} < K_1 (1 + \varrho + \varrho^2 + \ldots + \varrho^{n-2})$$
$$< \frac{K_1}{1 - \varrho}, \quad \ldots\ldots \quad (29)$$

so that on account of (18), (24), (25) we get

$$|u - \varphi(x)| < |u_1 - \varphi(x)| + |v_1 + v_2 + \dots|$$

$$< Ll + \frac{\mu K_1}{1 - \varrho} l^{\gamma+1}, \qquad \dots\dots \quad (30)$$

and similarly

$$|p - \varphi'(x)| < L\, l^{\gamma+\beta} + \frac{\mu K_1}{1 - \varrho} l^{\gamma+1/2},$$

$$|q - \theta(x)| < L\, l^{\gamma} + \frac{\mu K_1}{1 - \varrho} l^{\gamma} \qquad \dots\dots \quad (31)$$

Substituting (29) in (28) and remarking that $\Delta t < l$, we shall get an upper limit for $\frac{K_n}{K_{n-1}}$, and our object would be realised if this limit is equal to ϱ, for then we shall get successively $\frac{K_n}{K_{n-1}} < \varrho$ for all n.

We obtain thus a relation of the form

$$\varrho = \frac{A\, l^{\gamma}}{1 - \varrho} + B\, l^{\beta} + C\, l^{\gamma}, (\varrho < 1), \qquad \dots\dots \quad (32)$$

where A, B and C are polynomials in l.

Now (32) is an equation of the second degree, which gives us two roots between 0 and 1 if

$$Bl^{\beta} + Cl^{\gamma} < 1, \text{ and } (Bl^{\beta} + Cl^{\gamma} - 1)^2$$

$$- 4Al^{\gamma} \geqslant 0. \qquad \dots\dots \quad (33)$$

This gives us a maximum value l_1 of l.

We must now write the condition that the values of u_n, p_n, q_n, shall lie between $-N$ and $+N$. Let

$$M = \text{Max} \mid \varphi(x) \mid, \; M' = \text{Max} \mid \varphi'(x) \mid,$$
$$M'' = \text{Max} \;\; \theta(x), \qquad \ldots\ldots \quad (34)$$

so that M, M' and $M'' < N$. Then from (30) we have

$$\left.\begin{aligned} &M + Ll + \frac{\mu K_1}{1 - \varrho} l^{\gamma+1} < N, \\ &M' + Ll^{\gamma+\beta} + \frac{\mu K_1}{1 - \varrho} l^{\gamma+1/2} < N, \\ &M'' + Ll^{\gamma} + \frac{\mu K_1}{1 - \varrho} l^{\gamma} < N. \end{aligned}\right\} \quad \ldots \quad (35)$$

We must therefore take a value of ϱ such that it satisfies the equation (32) and the inequalities (35). Now these inequalities are satisfied by $l = 0$, and $\varrho = 0$, so that we shall certainly get in this way another upper limit l_2.

For the height l of our domain we then take the smaller of l_1 and l_2. Then the solution would be determined in a sufficiently small domain of height l, because the series

$$u_1 + \sum_n v_n, \; p_1 + \sum_n w_n, \; q_1 + \sum_n y_n$$

will be absolutely and uniformly convergent only when $t < l$. The terms of the series are continuous functions and therefore the sums will also be continuous.

The solution of the problem is then given by

$$u(x, t) = \lim_{n \to \infty} u_n(x, t), \qquad \ldots\ldots \quad (36)$$

where $u_n(x, t)$ is given by the relation (11).

The uniqueness of the solution can also be established as before for a sufficiently small domain.

CHAPTER V

FURTHER RESULTS FOR HYPERBOLIC EQUATIONS

5·1. An infinite system of non-linear Integro-differential Equations.

Consider the system

$$u_n(x) = \int_0^x g_n(x, y)\, F_n\{y; u_r(y); u'_r(y)\}\, dy, \qquad (1)$$

$(n = 1, 2, \ldots \infty)$, where $u'_r(y)$ denotes the derivative $\dfrac{du_r}{dy}$, and $u_r(y)$, $u'_r(y)$ represent the totality of the infinite sequences. We stipulate that the solutions $u_n(x)$ as well as their derivatives $u'_n(x)$ should belong to Hilbert space, i.e., that

$$\sum_n u_n^2(x) \text{ and } \sum_n u'^2(x)$$

should be convergent for all values of x in any finite interval.

The functions $g_n(x, y)$ and their derivatives $\dfrac{\partial g_n}{\partial x}$ are continuous for all x and y in $0 \leqslant y \leqslant x < \infty$;

and for all values of $x \geqslant 0$ and all real values of a_r, b_r belonging to Hilbert space, the functions $F_n(y; a_r; b_r)$ are uniformly continuous and satisfy Lipschitz condition:

$$\{F_n(y; a_r, b_r) - F_n(y; c_r, d_r)\}^2$$
$$\leqslant \alpha_n^2 \sum_{k=1}^{\infty} (a_k - c_k)^2 + \beta_n^2 \sum_{k=1}^{\infty} (b_k - d_k)^2, \quad \ldots\ldots \quad (2)$$

where α_n and β_n are positive constants.

Differentiating both sides of (1) *w. r.* to x, we have

$$u'_n(x) = g_n(x, x) F_n\{x; u_r(x); u'_r(x)\}$$
$$+ \int_0^x \frac{\partial g_n(x, y)}{\partial x} F_n\{y; u_r(y); u'_r(y)\} dy. \qquad (3)$$

Two cases now arise accordingly as $g_n(x, x)$ does or does not vanish identically for every n.

5·2. The case when $g_n(x, x) \equiv 0$.

Writing $v_n(x) = u'_n(x)$, $(n = 1, 2, \ldots \infty)$, we get from (1) and (3) § 5.1 the doubly infinite system of integral equations

$$u_n(x) = \int_0^x g_n(x, y) F_n\{y; u_r(y); v_r(y)\} dy, \qquad (1)$$

$$v_n(x) = \int_0^x \frac{\partial g_n}{\partial x}(x, y) F_n\{y; u_r(y); v_r(y)\} dy, \qquad (2)$$

$(n = 1, 2, \ldots \infty)$.

We solve this doubly infinite system by the usual method of successive approximations, and write for this purpose

$$\left.\begin{aligned} u_n^{(0)}(x) &= 0, \\ u_n^{(m)}(x) &= \int_0^x g_n(x, y)\, F_n\{y;\, u_r^{(m-1)}(y);\, v_r^{(m-1)}(y)\}\, dy; \end{aligned}\right\} \quad (3)$$

and

$$\left.\begin{aligned} v_n^{(0)}(x) &= 0, \\ v_n^{(m)}(x) &= \int_0^x \frac{\partial g_n}{\partial x}(x, y)\, F_n\{y;\, u_r^{(m-1)}(y);\, v_r^{(m-1)}(y)\}\, dy \end{aligned}\right\} \quad (4)$$

We shall prove that these approximations are uniformly convergent for all x, so that

$$\lim_{n \to \infty} u_n^{(m)}(x) = u_n(x);\ \lim_{n \to \infty} v_n^{(m)}(x) = v_n(x), \quad \ldots \quad (5)$$

and that

$$\sum_n u_n^2(x) \text{ and } \sum_n v_n^2(x) \quad \ldots\ldots \quad (6)$$

are convergent; further

$$\begin{aligned} &\lim_{m \to \infty} \sum_n \{u_n^{(m)}(x) - u_n(x)\}^2 \\ &= \lim_{m \to \infty} \sum_n \{v_n^{(m)}(x) - v_n(x)\}^2 = 0, \quad \ldots\ldots \quad (7) \end{aligned}$$

uniformly *w. r.* to x.

To establish these results it is necessary to impose more stringent restrictions on the given functions than in Chapter II. Thus we assume that

constants a, b, μ and ν exist (a, $b > 0$; $\mu, \nu \geqslant 0$,) such that

$$\left.\begin{array}{l} \sum_n \left\{ \int_0^x g_n(x, y)\, F_n(y; 0; 0)\, dy \right\}^2 \\ \sum_n \left\{ \int_0^x \frac{\partial g_n}{\partial x}(x, y)\, F_n(y; 0; 0)\, dy \right\}^2 \end{array}\right\} < ax^{\mu}; \qquad (8)$$

and

$$\left.\begin{array}{l} \sum_n \left\{ \alpha_n^2 \int_0^x g_n^2(x, y)\, dy \right\} \\ \sum_n \left\{ \beta_n^2 \int_0^x g_n^2(x, y)\, dy \right\} \\ \sum_n \left\{ \alpha_n^2 \int_0^x \left(\frac{\partial}{\partial x} g_n(x, y) \right)^2 dy \right\} \\ \sum_n \left\{ \beta_n^2 \int_0^x \left(\frac{\partial}{\partial x} g_n(x, y) \right)^2 dy \right\} \end{array}\right\} < bx^{\nu}. \qquad \ldots\ldots \quad (9)$$

Then from (3) we have

$$u_n^{(m+1)}(x) - u_n^{(m)}(x) = \int_0^x g_n(x, y)\, [F_n\{y; u_r^{(m)}(y);$$
$$v_r^{(m)}(y)\} - F_n\{y; u_r^{(m-1)}(y); v_r^{(m-1)}(y)\}]\, dy,$$

so that

$$\{u_n^{(m+1)}(x) - u_n^{(m)}(x)\}^2$$
$$\leqslant \int_0^x g_n^2(x, y)\, dy \,.\int_0^x [\alpha_n^2 \sum_k \{u_k^{(m)}(y) - u_k^{(m-1)}(y)\}^2$$
$$+ \beta_n^2 \sum_k \{v_k^{(m)}(y) - v_k^{(m-1)}(y)\}^2]\, dy. \qquad \ldots\ldots \quad (10)$$

Summing over n and taking account of (9) we obtain:

$$\sum_n \{u_n^{(m+1)}(x) - u_n^{(m)}(x)\}^2$$
$$\leqslant bx^\nu \int_0^x \sum_n [\{u_n^{(m)}(y) - u_n^{(m-1)}(y)\}^2$$
$$+ \{v_n^{(m)}(y) - v_n^{(m-1)}(y)\}^2]\, dy. \qquad \ldots\ldots \quad (11)$$

Similarly from (4) we get

$$\sum_n \{v_n^{(m+1)}(x) - v_n^{(m)}(x)\}^2$$
$$\leqslant bx^\nu \int_0^x \sum_n [\{u_n^{(m)}(y) - u_n^{(m-1)}(y)\}^2$$
$$+ \{v_n^{(m)}(y) - v_n^{(m-1)}(y)\}^2]\, dy. \qquad \ldots\ldots \quad (12)$$

We write

$$w_{m+1}(x) = \sum_n \{u_n^{(m+1)}(x) - u_n^{(m)}(x)\}^2$$
$$+ \sum_n \{v_n^{(m+1)}(x) - v_n^{(m)}\}^2, \qquad \ldots\ldots \quad (13)$$

and obtain from (11) and (12)

$$w_{m+1}(x) \leqslant 2\, bx^\nu \int_0^x w_m(y)\, dy. \qquad \ldots\ldots \quad (14)$$

Hence

$$w_1(x) = \sum_n \{u_n^{(1)}(x)\}^2 + \sum_n \{v_n^{(1)}(x)\}^2$$
$$\leqslant 2\, a\, x^\mu, \text{ by (8)}.$$

Therefore from (14) we get

$$w_2(x) \leqslant 2bx^\nu \int_0^x 2ay^\mu\, dy = \frac{4abx^{\mu+\nu+1}}{\mu+1}.$$

Similarly

$$w_3(x) \leqslant 2bx^\nu \int_0^x \frac{4ab}{\mu+1} \cdot y^{\mu+\nu+1}\, dy$$

$$\leqslant \frac{8ab^2}{\mu+1} \cdot \frac{x^{\mu+2\nu+2}}{\mu+\nu+2} = \frac{2a \cdot (2b)^2 \cdot x^{2(\nu+1)+\mu}}{(\mu+1)(\mu+1+\nu+1)}.$$

In general, we get by induction for all $m \geqslant 1$;

$$w_m(x) \leqslant \frac{2a \cdot (2b)^{m-1}\, x^{(m-1)(\nu+1)+\mu}}{(\mu+1)(\mu+1+\nu+1)\ldots\{\mu+1+(m-2)(\nu+1)\}}$$

$$\leqslant \frac{2a\, x^\mu \cdot (2b\, x^{\nu+1})^{m-1}}{(m-1)!}, \quad \ldots\ldots \quad (15)$$

since $\mu + 1 + (m-2)(\nu+1) \geqslant 1 + m - 2$
$= m - 1.$

But from (10) and (13) we obtain

$$|u_n^{(m+1)}(x) - u_n^{(m)}(x)| \leqslant \left\{ bx^\nu \int_0^x w'_m(y)\, dy \right\}^{1/2}$$

$$\leqslant \left\{ \frac{a\, x^\mu\, 2^m\, b^m\, x^{\nu m}}{m!} \right\}^{1/2}. \quad (16)$$

The series $\sum\limits_{m=0}^{\infty} (u_n^{(m+1)}(x) - u_n^{(m)}(x))$ and similarly the series $\sum\limits_{m=0}^{\infty} (v_n^{(m+1)}(x) - v_n^{(m)}(x))$ are therefore absolutely and uniformly convergent in any finite interval of x. Hence

$$\lim_{m\to\infty} u_n^{(m)}(x) = u_n(x), \lim_{m\to\infty} v_n^{(m)}(x) = v_n(x),$$

uniformly *w. r.* to x. Thus (5) is established.

To prove (6) we write

$$
{}_pR_m(x) = \sum_n \{u_n^{(m+p)}(x) - u_n^{(m)}(x)\}^2 + \sum_n \{v_n^{(m+p)}(x) - v_n^{(m)}(x)\}^2. \qquad (17)
$$

Then we have from (13);

$$
\begin{aligned}
{}_pR_0(x) &\leqslant 2\sum_n [(u_n^{(1)}(x))^2 + (v_n^{(1)}(x))^2] \\
&\quad + 2\sum_n [(u_n^{(p)}(x) - u_n^{(1)}(x))^2 + (v_n^{(p)}(x) - v_n^{(1)}(x))]^2, \\
&\leqslant 2\,w_1(x) + 4b\,x^\nu \int_0^x {}_{p-1}R_0(y)\,dy \\
&\leqslant 4\,a\,x^\mu + 4b\,x^\nu \int_0^x {}_{p-1}R_0(y)\,dy. \qquad (18)
\end{aligned}
$$

Repeating the inequality (18), we can show by induction that

$$
\begin{aligned}
{}_pR_0(x) &\leqslant 4a\,x^\mu \Big\{ 1 + \frac{4b\,x^{\nu+1}}{\mu+1} + \frac{(4b\,x^{\nu+1})^2}{(\mu+1)(\mu+1+\nu+1)} \\
&\quad + \dots + \frac{(4b\,x^{\nu+1})^{p-1}}{(\mu+1)(\mu+1+\nu+1)\dots[\mu+1+(p-2)(\nu+1)]} \Big\} \\
&\leqslant 4a\,x^\mu \left\{ 1 + \frac{4b\,x^{\nu+1}}{1} + \dots + \frac{(4b\,x^{\nu+1})^{p-1}}{(p-1)\,!} \right\} \\
&< 4ax^\mu\, e^{4bx^{\nu+1}}. \qquad \dots\dots \quad (19)
\end{aligned}
$$

(6) is now obvious, because the right side of (19) is uniformly bounded in any finite interval of x, and is independent of p.

To prove (7), we must show that for all p

$$\lim_{m \to \infty} {}_pR_m(x) = 0 \qquad \ldots\ldots \quad (20)$$

uniformly *w. r.* to x.

Now from (17) we obtain

$${}_pR_m(x) \leqslant 2b\, x^\nu \int_0^x {}_pR_{m-1}(y)\, dy \qquad \ldots\ldots \quad (21)$$

But from (19) we have

$${}_pR_0(x) \leqslant 4a\, x^\mu\, e^{4b\, x^{\nu+1}},$$

so that

$${}_pR_1(x) \leqslant 2b\, x^\nu \int_0^x 4a\, y^\mu\, e^{4b\, y^{\nu+1}}\, dy$$

$$< 4a \,.\, 2b \,.\, x^\nu\, e^{4b\, x^{\nu+1}} \int_0^x y^\mu\, dy$$

$$< \frac{4a \,.\, 2b \,.\, x^{\mu+\nu+1}\, e^{4b\, x^{\nu+1}}}{\mu+1}.$$

Thus, in general for all $m \geqslant 1$ we get

$${}_pR_m(x)$$

$$\leqslant \frac{4a \,.\, (2b\, x^{\nu+1})^m\, e^{4b\, x^{\nu+1}}}{(\mu+1)(\mu+1+\nu+1)\ldots[\mu+1+(m-1)(\nu+1)]}$$

$$< \frac{4a\, (2b\, x^{\nu+1})^m\, e^{4b\, x^{\nu+1}}}{m\,!}. \qquad \ldots\ldots \quad (22)$$

This proves (20) and consequently (7).

Proceeding to the limit $m \to \infty$ in (3) and (4), we obtain $u_n(x), v_n(x), (n = 1, 2, \ldots, \infty)$, which

are solutions of (1) and (2), and consequently $u_n(x)$ is the solution of (1) § 5.1.

We shall now prove that $u_n(x)$ is the only solution of its kind. If possible, suppose that $\bar{u}_n(x)$ $\left[\text{with } \bar{v}_n(x) = \frac{d\bar{u}_n}{dx}\right]$ is another solution of the equation (1) § 5.1., such that

$$\sum_n \{\bar{u}_n(x)\}^2 \text{ and } \sum_n \{\bar{v}_n(x)\}^2$$

are uniformly convergent.

Let for all $m \geqslant 1$:

$$\left.\begin{aligned} A_m &= \sum_n \{u_n^{(m)}(x) - u_n(x)\}^2 \\ &\quad + \sum_n \{v_n^{(m)}(x) - v_n(x)\}^2 \\ B_m &= \sum_n \{u_n^{(m)}(x) - \bar{u}_n(x)\}^2 \\ &\quad + \sum_n \{v_n^{(m)}(x) - \bar{v}_n(x)\}^2 \end{aligned}\right\} \quad \ldots \quad (23)$$

Then we find

$$\sum_n \{u_n(x) - \bar{u}_n(x)\}^2 + \sum_n \{v_n(x) - \bar{v}_n(x)\}^2 \leqslant 2(A_m + B_m) \quad \ldots\ldots \quad (24)$$

It is not difficult to show on the same lines as in (22) that

$$\lim_{m \to \infty} B_m = 0. \quad \ldots\ldots \quad (25)$$

From (7) we know further that $\lim_{m \to \infty} A_m = 0$. Consequently, we conclude from (24) that

$$\sum_n \{u_n(x) - \bar{u}_n(x)\}^2 \equiv 0, \quad \sum_n \{v_n(x) - \bar{v}_n(x)\}^2 \equiv 0. \qquad \ldots\ldots \quad (26)$$

This shows that $\bar{u}_n(x) \equiv u_n(x)$ and $v_n(x) \equiv v_n(x)$ for every n, so that the solution $u_n(x)$ of (1) § 5.1 is unique.

5·3. The case when $g_n(x, x) \not\equiv 0$.

In this case the doubly infinite system of non-linear integral equation (1) and (3) § 5.1 can be written, when $v_n(x)$ is put for $u'_n(x)$,

$$\left.\begin{aligned} u_n(x) &= \int_0^x g_n(x, y)\, F_n\{y; u_r(y); v_r(y)\}\, dy, \\ v_n(x) &= g_n(x, x)\, F_n\{x; u_r(x); v_r(x) \\ &\quad + \int_0^x \frac{\partial}{\partial x} g_n(x, x)\, F_n\{y; u_r(y); v_r(y)\}\, dy. \end{aligned}\right\} \qquad (1)$$

In addition to the hypothesis (2) § 5.1 and (8), (9) § 5.2, we must now assume that absolute constants $c, d > 0$ and $\gamma, \delta \geqslant 0$ exist such that

$$\sum_n \{g_n(x, x)\, F_n(x; 0; 0)\}^2 < c x^{\gamma}; \qquad \ldots\ldots \quad (2)$$

$$\left.\begin{aligned} &\sum_n \{\alpha_n^2 g_n^2(x, x)\} \\ &\sum_n \{\beta_n^2 g_n^2(x, x)\} \end{aligned}\right\} < d x^{\delta}. \qquad \ldots\ldots \quad (3)$$

We take again the successive approximations

$$\left.\begin{aligned} u_n^{(0)}(x) &= 0,\ v_n^{(0)}(x) = 0, \\ u_n^{(m)}(x) &= \int_0^x g_n(x, y)\, F_n\{y;\, u_r^{(m-1)}(y);\, v_r^{(m-1)}(y)\}\, dy, \\ v_n^{(m)}(x) &= g_n(x, x)\, F_n\{x;\, u_r^{(m-1)}(x);\, v_r^{(m-1)}(x)\} \\ &\quad + \int_0^r \frac{\partial g_n}{\partial x}(x, y)\, F_n\{y;\, u_r^{(m-1)}(y);\, v_r^{(m-1)}(y)\}\, dy. \end{aligned}\right\} \qquad \ldots\ldots \quad (4)$$

We can show that *if x lies in a restricted interval* then for all $n \geqslant 1$:

$$\lim_{m \to \infty} u_n^{(m)}(x) = u_n(x);\ \lim_{m \to \infty} v_n^{(m)}(x) = v_n(x). \quad \ldots \quad (5)$$

uniformly *w. r.* to x, and that

$$\lim_{m \to \infty} \sum_n \{u_n^{(m)}(x) - u_n(x)\}^2 = \lim_{m \to \infty} \sum_n \{v_n^{(m)}(x) - v_n(x)\}^2 = 0 \qquad \ldots\ldots \quad (6)$$

uniformly *w. r.* to x. The relation (6) implies that $u_n(x)$ and $v_n(x)$ belong to Hilbert space.

We shall give only a brief outline of the proof: the details are as in § 5.2.

Let

$$_pR_m(x) = \sum_n [\{u_n^{(m+p)}(x) - u_n^{(m)}(x)\}^2 + \{v_n^{(m+p)}(x) - v_n^{(m)}(x)\}^2]. \qquad \ldots\ldots \quad (7)$$

Then from (4) we obtain

$$
\begin{aligned}
{}_pR_m(x) &\leqslant bx^\nu \int_0^x {}_pR_{m-1}(y)\,dy + 2dx^\delta\,{}_pR_{m-1}(x) \\
&\qquad\qquad + 2bx^\nu \int_0^x {}_pR_{m-1}(y)\,dy \\
&\leqslant 2dx^\delta\,{}_pR_{m-1}(x) + 3bx^\nu \int_0^x {}_pR_{m-1}(y)\,dy \\
&\leqslant (2dx^\delta + 3bx^\nu\,\mathrm{I})\,{}_pR_{m-1}(x), \qquad \ldots\ldots \quad (8)
\end{aligned}
$$

where I stands for the integral operator $\int_0^x dy$. Thus

$$
{}_pR_m(x) \leqslant (2dx^\delta + 3bx^\nu\,\mathrm{I})^m\,{}_pR_0(x) \qquad \ldots\ldots \quad (9)
$$

Now we find analogous to (19) that

$$
\begin{aligned}
{}_pR_0(x) &\leqslant \phi(x) + (3dx^\delta + 5bx^\nu\,\mathrm{I})\,{}_{p-1}R_0, \qquad \ldots \quad (10) \\
&\leqslant \sum_{k=0}^{p-1}(3dx^\delta + 5bx^\nu\,\mathrm{I})^k\,\phi(x) + (3dx^\delta + 5bx^\nu\,\mathrm{I})^p\,{}_1R_0(x),
\end{aligned}
$$

where $\qquad \phi(x) = 8ax^\mu + 6cx^\gamma. \qquad \ldots\ldots \quad (11)$

Similarly

$$
{}_1R_0(x) \leqslant 2cx^\gamma + 3ax^\mu. \qquad \ldots\ldots \quad (12)
$$

We assume x to be so small that

$$
x < \left(\frac{1}{3d}\right)^{1/\delta}. \qquad \ldots\ldots \quad (13)
$$

It can be shown that on account of (12), the last term on the right of (10) tends to zero as $p \to \infty$,

uniformly *w. r.* to x provided x satisfies (13), and that the sum $\sum_{k=0}^{p-1} (3dx^{\delta} + 5bx^{\nu} \mathrm{I})^k \phi(x)$ remains bounded. Hence ${}_pR_0(x)$ is uniformly bounded for all p. Therefore from (8) we get

$$\lim_{m \to \infty} {}_pR_m = 0 \quad \ldots\ldots \quad (14)$$

for all p.

This shows that for the restricted interval (13) the limits

$$\lim_{m \to \infty} u_n^{(m)}(x) = u_n(x), \lim_{m \to \infty} v_n^{(m)}(x) = v_n(x) \quad (15)$$

exist for every n. We see also that $\sum_n \{u_n(x)\}^2$ and $\sum_n \{v_n(x)\}^2$ are convergent.

These $u_n(x)$ and $v_n(x)$ are solutions of the doubly infinite system (1), (2); and it can be proved as in § 5.2 that the solutions are unique.

The system of integral equations

$$u_n(x) = \int_0^x g_n(x, y) F_n \{y; u_r(y)\} dy \quad \ldots\ldots \quad (16)$$

is evidently much easier to deal with than the system of integro-differential equations above. It can readily be seen that the system (16) has unique

solutions belonging to Hilbert space[1].

5·4. Non-linear hyperbolic equation.

In Chapter II we considered the equation

$$\frac{\partial}{\partial x}\left\{\varrho(x)\frac{\partial u}{\partial x}\right\} - \frac{\partial^2 u}{\partial t^2} = P(x, t, u),$$

where $P(x, t, u)$ was supposed to be an analytic function which could be represented in a series $\sum_{r=2}^{\infty} p_r(x, t)\, u^r$. The method was evidently not applicable if P contained $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ also. But the system of integro-differential equations studied in § 5.3 makes it possible to prove the existence and uniqueness of the solution of the equation

$$\frac{\partial}{\partial x}\left\{\varrho(x)\frac{\partial u}{\partial x}\right\} - \frac{\partial^2 u}{\partial t^2} = P(x, t, u, p, q), \quad \dots \quad (1)$$

$$\left[p = \frac{\partial u}{\partial x},\ q = \frac{\partial u}{\partial t}\right]$$

for the boundary conditions

$$u(0, t) = u(\pi, t) = 0 \text{ for all } t \text{ in } 0 \leqslant t \leqslant T, \qquad (2)$$

[1]For details of § 5.1—5.3, cf. S. M. Sundaram: "On an infinite system of non-linear integro-differential equations." Proc. Ind. Acad. Sc. *9* (1939) p. 410—418; "On an infinite system of non-linear integral equations." Ibid, *8* (1938) p. 238—242.

and

$$u(x, 0) = f_1(x), \frac{\partial u}{\partial t}(x, 0) = f_2(x) \text{ in } 0 \leqslant x \leqslant \pi \quad (3)$$

The function $\varrho(x)$ is essentially positive, continuous and bounded together with its first and second derivatives in $0 \leqslant x \leqslant \pi$:

$$\left. \begin{aligned} 0 < m \leqslant \varrho(x) \leqslant M, \\ |\varrho'(x)| \leqslant M, |\varrho''(x)| \leqslant M. \end{aligned} \right\} \quad (4)$$

$P(x, t, u, p, q)$ is continuous *w. r.* to all the variables and

$$P(0, t, 0, a, 0) = P(\pi, t, 0, a, 0) = 0, \quad (5)$$

for all real a and all $t \geqslant 0$, so that P can be expanded in a series of Sturm-Liouville eigen-functions $\phi_n(x)$, [with $w_n(t)$ as in (11) below]:

$$\left. \begin{aligned} P(x, t, u, p, q) &= \sum_n F_n \{t; w_r(t); w_r'(t)\} \phi_n(x), \\ F_n\{t; w_r(t); w_r'(t)\} &= \int_0^\pi P(r, t, u, p, q) \phi_n(x) dx. \end{aligned} \right\} \quad (6)$$

We assume further that P is Lipschitzian *w. r.* to u, p, q so that

$$\{P(x, t, u, p, q) - P(x, t, \bar{u}, \bar{p}, \bar{q})\}^2$$
$$\leqslant K_1 \{(u - \bar{u})^2 + (p - \bar{p})^2 + (q - \bar{q})^2\} \quad (7)$$

where K_1 is an absolute constant > 0.

The given functions $f_1(x)$ and $f_2(x)$ are also

taken to be capable of being expanded in a series of $\varphi_n(x)$;

$$\left.\begin{aligned} f_1(x) &= \sum_n c_n \phi_n(x), \\ f_2(x) &= \sum_n d_n \phi_n(x), \end{aligned}\right\} \quad (8)$$

We assume further that $f_1'(x)$ and $f_2(x)$ are integrable L^2, and write

$$\sum_n \lambda_n c_n^2 + \sum d_n^2 = K_2, \qquad \ldots\ldots \quad (9)$$

and

$$\sum_n F_n^2(t; 0; 0) = \int_0^\pi P^2(x, t, 0, 0, 0)\, dx < K_3 \quad (10)$$

where K_2 and K_3 are absolute constants.

For the solution of the equation (1), we write

$$u(x, t) = \sum_n w_n(t)\, \phi_n(x), \qquad \ldots \quad (11)$$

so that the conditions (3) and (8) will be satisfied if for all $n \geqslant 1$ we have

$$w_n(0) = c_n,\ w_n'(0) = d_n. \quad (12)$$

Now, if $\bar{u}(x, t)$ is another function of the form (11), we get

$$\sum_n [F_n\{t; w_r(t); w_r'(t)\} - F_n\{t; \bar{w}_r(t)\, \bar{w}_r'(t)\}]^2$$
$$= \int_0^\pi \{P(x, t, u, p, q) - P(x, t, \bar{u}, \bar{p}, \bar{q})\}^2\, dx$$
$$\leqslant K_1 \left\{ \int_0^\pi (u-\bar{u})^2 dx + \int_0^\sigma (p-\bar{p})^2 dx + \int_0^\pi (q-\bar{q})^2 dx \right\}$$

$$\leqslant \left(\frac{K_1}{\lambda_1} + \frac{K_1}{m}\right) \sum_n \lambda_n (w_n - \overline{w}_n)^2 + K_1 \sum_n (w'_n - \overline{w}'_n)^2$$

$$\leqslant K_4 \left\{\sum_n \lambda_n (w_n - \overline{w}_n)^2 + \sum_n (w'_n - \overline{w}'_n)^2\right\} \qquad (13)$$

where $$K_4 = \max\left(K_1, \frac{K_1}{\lambda_1} + \frac{K_1}{m}\right).$$

Substituting (6), (11) in (1), and equating the Fourier Coefficients on both sides, we obtain for all $n \geqslant 1$:

$$\frac{d^2 u_n}{dt^2} + \lambda_n u_n = - F_n \{t; w_r(t); w'_r(t)\} \qquad (14)$$

The solution of (14) satisfying the initial conditions (12) is given by

$$w_n(t) = c_n \cos \sqrt{\lambda_n}\, t + \frac{d_n}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n}\, t$$

$$- \frac{1}{\sqrt{\lambda_n}} \int_0^t \sin \sqrt{\lambda_n}(t-y) \times F_n \{y; w_r(y); w'_r(y)\}\, dy, \quad (15)$$

$(n = 1, 2, \ldots. \infty)$.

Differentiating (15) *w. r.* to t we obtain

$$w'_n(t) = - \sqrt{\lambda_n}\, c_n \sin \sqrt{\lambda_n}\, t + d_n \cos \sqrt{\lambda_n}\, t$$

$$- \int_0^t \cos \sqrt{\lambda_n}(t-y) \times F_n \{y; w_r(y); w'_r(y)\}\, dy, \quad (16)$$

$(n = 1, 2, \ldots, \infty)$.

We set

$$u_n(t) = \sqrt{\lambda_n}\, w_n(t),\; v_n(t) = w_n'(t), \quad \ldots\ldots \quad (17)$$

and get from (15) and (16) the doubly infinite system of non-linear integral equations

$$\left.\begin{aligned}
u_n(t) &= \sqrt{\lambda_n}\, c_n \cos\sqrt{\lambda_n}\, t + d_n \sin\sqrt{\lambda_n}\, t \\
&\quad - \int_0^t \sin\sqrt{\lambda_m}(t-y) \times F_n\Big\{y; \frac{u_r(y)}{\sqrt{\lambda_r}}; v_r(y)\Big\}\, dy; \\
v_n(t) &= -\sqrt{\lambda_n}\, c_n \sin\sqrt{\lambda_n}\, t + d_n \cos\sqrt{\lambda_n}\, t \\
&\quad - \int_0^t \cos\sqrt{\lambda_n}(t-y) \times F_n\Big\{y; \frac{u_r(y)}{\sqrt{\lambda_r}}; v_r(y)\Big\}\, dy.
\end{aligned}\right\} (18)$$

$$(n = 1, 2, \ldots, \infty).$$

From § 5.3 we see then that one and only one solution of the doubly infinite system (18) exists for a finite interval of t, such that $\sum\limits_n u_n^2(t) = \sum\limits_n \lambda_n w_n^2(t)$ and $\sum\limits_n v_n^2(t) = \sum\limits_n \{w_n'(t)\}^2$ are uniformly convergent, and consequently $\dfrac{\partial u}{\partial x}(x, t)$ and $\dfrac{\partial u}{\partial t}(x, t)$ are integrable L^2 uniformly in t.

Thus $u(x, t) = \sum\limits_n w_n(t)\, \phi_n(x)$ is the required solution of (1) satisfying the boundary conditions (2) and (3).

If $P(x, t, u, p, q)$ is defined in the bounded

five-dimensional domain

$$0 \leqslant x \leqslant \pi;\ 0 \leqslant t \leqslant T;\ |u|, |p|, |q| \leqslant N, \quad \ldots \quad (19)$$

and if P_x, P_u, P_p, P_q exist and are bounded, and if $f_1''(x)$ and $f_2''(x)$ exist and are integrable L^2 so that $\sum_n \lambda_n^2 c_n^2$ and $\sum_n \lambda_n d_n^2$ are convergent, then the equation (1) can be solved uniquely in the form (11) so that for sufficiently small values of t, the series $\sum_n \lambda_n^2 w_n^2(t)$ and $\sum_n \{w_n'(t)\}^2$ are uniformly convergent.

It is not difficult to see that the method can be readily extended to the following cases:—

(*a*) Instead of the boundary condition (2), we can have one of the following boundary conditions;

$$\left.\begin{aligned} &\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0, \\ &u(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0, \\ &u(0, t) = u(\pi, t);\ h(0)\frac{\partial u}{\partial x}(0, t) \\ &\qquad\qquad = h(\pi)\frac{\partial u}{\partial x}(\pi, t) \end{aligned}\right\} \quad (20)$$

(*b*) Instead of the differential equation (1), we

can have a finite system of equations

$$\frac{\partial}{\partial x}\left\{\varrho_i(x)\frac{\partial u_i}{\partial x}\right\} - \frac{\partial^2 u_i}{\partial t^2}$$
$$= P_i(x, t, u_1, \ldots, u_m, p_1, \ldots p_m, q_1, \ldots q_m)$$
$$(i = 1, 2, \ldots, m). \qquad \ldots\ldots \quad (21)$$

(*c*) Instead of two independent variables x, t, we can have more independent variables $x,_1$ $x_2, \ldots, x_m$, t so that equation (1) becomes

$$\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{\varrho(x_1, \ldots x_m)\frac{\partial u}{\partial x_i}\right\} - \frac{\partial^2 u}{\partial t^2}$$
$$= P(x_1, \ldots, x_m, t, u, p_1, \ldots, p_m\, q) \qquad \ldots\ldots \quad (22)$$

where $$p_i = \frac{\partial u}{\partial x_2}.$$

It is obvious that in the extensions (*b*) and (*c*) the boundary values would have to be generalised in a corresponding manner.

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